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A METHOD FOR DEMONSTRATING MISSILE RELIABILITY CONCURRENTLY WITH DEVELOPMENT

ROBERT GEORGE HARRIS

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CONCURRENTLY WITH DEVELOPMENT

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Robert G. Harris

A METHOD FOR DEMONSTRATING MISSILE RELIABILITY CONCURRENTLY WITH DEVELOPMENT

by

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Major, United States Marine Corps

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

United States Naval Postgraduate School
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MASTER OF SCIENCE
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ABSTRACT

This thesis presents a method for indicating, during the developmental phase, reliability levels attained and the probability of achieving a specified reliability goal. The results from developmental tests are used, as contrasted to the more prevalent practice of testing for reliability demonstration at, or near the conclusion of development. A significant advantage is thereby available to management and the designer by reducing time and expenditure in the decision making process. Technically, this procedure is based on a principle of a posteriori probability. This requires an assumption of an a priori probability distribution which is then modified by test results, according to Bayes' Theorem, to yield an a posteriori probability distribution.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS

1. Introduction

It is becoming increasingly prevalent for a reliability figure to be specified for a system prior to the design and development of the equipment. This means that the developer must build the equipment so that its probability of satisfactory operation be at least as high as the specified goal.

Specification of a reliability figure as a goal implies that the attainment of this figure must be demonstrated. In many instances, this demonstration of reliability is a contractual obligation.

In order to demonstrate reliability by conventional sampling procedures subsequent to development, extensive testing at considerable cost in time, money, and materials is generally required. This cost may not be economically feasible for such expensive programs as ballistic missiles.

The delaying of the determination of the reliability level until after development would extend the time required prior to the inception of production and could negate a majority of the past development effort if the reliability goal is not achieved. A demonstrated reliability level falling short of the goal will necessitate a re-examination of the design, require a re-development, and follow with another testing period. Such a long feedback loop for decision making is intolerable. Consequently, a product less reliable than desired may be accepted.

A method that would progressively demonstrate the reliability level attained, concurrently with the developmental program, is desired. Such a method could utilize the results of developmental testing, and thereby reduce the costs of testing solely for the purpose of measuring reliability.

2. Probability, a priori and a posteriori

A definition of probability can be given in several ways but the following classical definition of probability will suffice at this point. The probability that a specific unit possesses a certain attribute is defined to be that proportion, q, of the units possessing the attribute within the entire population of units. Thus, from a knowledge of the composition of the population, it is possible to compute the probability that a particular composition will occur in a single randomly drawn sample. Such probabilites, that is, probabilities determined from the classical definition, are called "a priori" probabilities.

A somewhat reverse situation exists in the case of "a posteriori" probabilities. All that is known is the composition of a randomly drawn sample. The technique used in this paper assumes a probability distribution for the unknown q based upon experience and sound engineering judgment. This assumed distribution is referred to as an a priori probability distribution. A conditional distribution of q given the outcome of experiments (i.e., test results or sample values) is then computed. This conditional probability distribution of q is known as the a posteriori distribution of q, and probabilities computed from this distribution are called a posteriori probabilities. The technique is sometimes referred to as a Bayes' principle since Bayes' formula is used in computing the conditional probability distributions.

3. Use of Developmental Testing

For eaoh level of reliability there is a minimum number of successful tests required. This number varies with the number of test failures experienced as well as the confidence level associated with the statement of reliability. It is also evident that the number of successful test results needed depends on the a priori assumption. When an optimistic a priori distribution is assumed, the number of successful tests required is reduced accordingly. This may appear at first sight to be a very unsafe procedure. However, it will be shown by the operating characteristics of the procedure that there is a low probability of achieving a successful count unless the actual system reliability is as good as or better than the required level.

The purpose of this paper is to propose a method that uses the knowledge gained from developmental testing to indicate the reliability of the equipment at each stage of its development. It also will make it possible to ascertain when, and if, a specified system reliability has a reasonably good chance of being achieved. From this information, the criteria for re-examination of the current design can be formulated. If the reliability of the current design is low or the probability of this design reaching the specified reliability level is low, redesign or modification is indicated. By having this reliability information concurrently with each step in development, the designer will be able to improve his product at an earlier stage, thereby shortening the feedback loop and ensuring earlier achievement of the desired goals. As a consequence of better feedback, the number of tests required will probably be reduced.

Clearly, a reduction in tests is a savings in money, time, and material. There is a variability in the amount of testing required depending on the actual reliability of the equipment. The higher the actual reliability, the smaller the amount of testing required.

4. Comments Concerning Sequential Sampling

It has been suggested that standard sequential sampling plans be used as a method of reducing the sample size required to demonstrate a specified reliability level. The method proposed here, while being of a sequential nature, differs in several important ways. Sequential analysis, such as that which might follow development, includes criteria for accepting or rejecting the system. The proposed method can be used concurrent with development and does not "reject", but indicates the reliability level attained and the probability of achieving the specified reliability goal.

The sequential analysis technique appears to be best applied to product improvement programs. Unfortunately, several missile manufacturers, after completing development, have found that they have produced an unreliable missile. The purpose of the proposed method is to avoid this contingency by providing management and the designer with reliability "milestones" during development. A very important advantage of the method is that it does not require equal amounts of testing on all components. Extra testing is concentrated on components indicating inferior reliability. Subsequent test results of these components may, by this procedure, be integrated with test results of other components in the system to demonstrate achievement of the overall system reliability.

5. The Distribution Function

Let p be the proportion of the population of systems under consideration that will be successful, then $q = 1 - p$ will be the proportion that will fail.

For this population of systems, there exists some single value of q. Since this value is unknown, an a priori distribution of q values is assumed. This is equivalent to treating the failure proportion as a random variable, Q.[1]

Assume that a typical a priori distribution, $f_0(q)$, of Q is the Beta distribution characterized by two parameters, A and B, where A is associated with successes and B with failures. Then:

$$f_0(q) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} p^A q^B \tag{1}$$

$$= \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} (1-q)^A q^B \qquad \begin{matrix} A>0 \\ B>-1 \end{matrix}$$

with its mean given by

$$\mu_q = \frac{B+1}{A+B+2}$$

its mode by

$$\nu_q = \frac{B}{A+B}$$

and its variance by

$$\sigma_q^2 = \frac{(A+1)(B+1)}{(A+B+2)^2(A+B+3)}$$

See Appendix IV for mathematical derivation.

[1] A random variable may be defined to be a function which associates a real number with every possible outcome of an experiment according to some probability law. Conventionally, a random variable is designated by a capital letter and any value of this random variable by a small letter. In this case, Q is a continuous random variable defined on the interval $0 \leq q \leq 1$.

The Beta distribution is particularly adaptable to Bayes' Theorem since the a priori distribution function (a Beta distribution), when modified by experience (success and/or failure), results in an a posteriori distribution which is again a Beta distribution. Besides this desirable property, the various forms of the Beta distribution are sufficiently inclusive to represent realistic distributions of Q. This distribution varies from the uniform distribution where A = B = 0, to the extremely skewed distribution where $A \gg B$, $A \gg 1$.[1] When $A \gg B$ and $A \gg 1$, the Beta distribution can be approximated by the Gamma distribution. For a system whose reliability is high, the distribution of Q must necessarily be very skewed. To demonstrate such reliability, one should expect the number of successes to be large and much greater than the number of failures. Therefore, the Gamma distribution function becomes a realistic approximation to the Beta distribution and will be used. The benefits from this approximation will become more apparent in the subsequent sections.

If $A \gg B$ and $A \gg 1$, the values of the parameters of the distribution can be approximated as follows:

$$\mu_q \cong \frac{B+1}{A} \qquad (2)$$

$$\sigma_q \cong \frac{\sqrt{B+1}}{A}$$

[1] The symbol $\gg$ means "much greater than".

If S successes and F failures are obtained in N independent trials of a system whose failure probability q is a number from a population of Q assumed to have an a priori probability density $f_o(q)$, then a posteriori probability density $f_N(q)$ is given by:

$$f_N(q) = \frac{f_o(q)\left[C_S^N p^S q^F\right]}{\int_0^1 \left[f_o(q)\, C_S^N p^S q^F\right] dq} \tag{3}$$

which reduces to an expression identical with that for $f_o(q)$ in (1), except that A is replaced by A + S and B by B + F. In particular, if $S \gg F$, the mean and standard deviation of the a posteriori distribution, using (2), are approximately:

$$\mu_q \cong \frac{B+F+1}{A+S} \tag{4}$$

$$\sigma_q \cong \frac{\sqrt{B+F+1}}{A+S}$$

(See Appendices I through IV for the mathematical derivations)

6. Confidence and Reliability

Classically, the concept of confidence is based on results of observations made on elements of a population. It is a statement of the degree of assurance one has in a statement about a parameter of the population. In this paper, a priori "confidence" is an assumption in the form of an a priori probability. After observing the results of testing, this confidence is modified in an <u>a posteriori probability</u> and correspondingly an a posteriori confidence.

With each statement about q, the unreliability, there is an associated confidence. This confidence is improved as one gains more knowledge about the distribution of Q.

Let any fixed value of Q, say q_c, be the maximum allowable system failure probability. Then the probability of $Q < q_c$ can be evaluated by the integral:

$$(5) \qquad \int_0^{q_c} f_N(q)\,dq$$

This is the probability that the q of the system is included in the interval zero to q_c. This q is a number from a population of numbers distributed a priori by $f_o(q)$ and a posteriori by $f_N(q)$. This probability is the confidence, in view of the a priori probability distribution and test results, with which it is stated that the desired reliability has been attained. This incomplete Beta function (5) will be evaluated by an incomplete Gamma function using the same mean and standard deviation given by (4).

Let $z = Aq$

then
$$\underset{A \to \infty}{\text{Limit}}\ f_N(q)\,dq = \frac{e^{-z} z^B dz}{\Gamma(B+1)}$$

and

(6)
$$\alpha(z_c) \cong \int_0^{z_c} \frac{e^{-z} z^B dz}{\Gamma(B+1)}$$

In general

(7)
$$\text{Prob.}\{Q < q_c\} \cong \alpha(z_c)$$

Using the <u>Tables of the Incomplete Gamma Function</u>, it is possible to determine the value of z corresponding to a given B for different levels of confidence. (See Appendices VI and VII for details). In Figure (1), the plot of B vs. z is presented for 50%, 70%, 80%, 90%, and 95% confidence levels.

As an example of the use of this chart, assume that you have experienced 20 successes and 4 failures and wish to make a statement about the system's reliability with a 90% confidence. Entering the chart with $B = 4$, you read $z = Aq = 8$, from which $q = .4$. Therefore, you may state that the system has a reliability of .6 or better with the confidence that a statement of this sort will be correct 90 times out of 100.

7. Assumption of Initial Values

It should be noted that the proposed method permits a certain amount of subjectivity on the part of the experimenter. His selection of a particular a priori distribution function of Q reflects the degree of optimism with which he views the design. As was mentioned previously, any optimism should be the consequence of design experience and sound engineering judgment.

When a reliability goal has been set for the system, a specified value q_c is established. Consider the case when the mean, μq, of the distribution of Q is equal to q_c, and the distribution is spread so wide that the standard deviation is also equal to q_c.

According to (2), this requires that the initial values of the parameters must be:

$$A = 1/q_c \qquad A \gg 1$$
$$B = 0 \qquad A \gg B \tag{8}$$

This means that a priori there is about 70% assurance that $Q < q_c$.

In effect, this assumption credits the design with $\frac{1}{q_c}$ successes toward demonstration of the reliability goal. However, even with this advance "credit", the probability of an unreliable system reaching the goal by chance alone is quite small. Section 9 on the Operating Characteristics curve (O.C. curve) illustrates this point.

To further investigate the effect of assuming some initial value for A, consider $A = \frac{1}{2q_c}$ and $A = \frac{2}{q_c}$. The effects of these biases are simularly exhibited by the O. C. curve.

8. The Demonstration Walk

Now the question of how many system test successes are needed before there is reasonable assurance that the failure probability Q is less than some pre-assigned q_c can be answered.

Two different graphical presentations are found convenient to answer this question. With the number of failures versus the number of successes as coordinates, it is possible to plot a family of curves for either constant values of q_c or percent levels of confidence. These two types of presentations are illustrated by Figures 2-2.5 and 3-3.5 respectively.

As the experimenter records the test results on the q_c curve graphs (Figures 2-2.5), he performs a type of "random walk". Starting at the origin, the walk moves one unit upward for each success and one unit to the right for each failure. When the walk reaches one of the q_c lines, the system under development has demonstrated the reliability associated with this curve.

Referring to Figure 2, consider the following illustrative example. Assume that the assigned reliability goal is .80. This fixes the q_c at .20. Figure 2 shows that a minimum of 12 successes without failure is required to demonstrate the specified reliability goal. If only one failure is experienced before 12 successful tests have been conducted, the required number of successes increases to 20. Similarly, the number of successful test results for any number of failures can be determined.

Next, let one consider the second type of presentation as exemplified by Figures 3 through 3.5. In this type of chart, the family of curves for several percent levels of confidence are plotted for a given q_c value.

After the reliability goal has been established and the development program has begun, management will be interested in knowing how development is progressing with respect to the established goal. This type of chart provides a graphic display of the degree of confidence which can be ascribed to certain figures of reliability.

As an illustrative example, consider Figure 3. Assume the reliability goal is .80 or $q_c = .20$. Further assume that 10 successes and 1 failure have been experienced. From the chart, it is seen that there is more than 50% confidence of .80 reliability or better. If an additional 18 successes and another failure were subsequently recorded, the confidence would be increased to over 90%.

Next, consider the effect of letting $A = \frac{1}{q_c}$ and $B = 0$. Since $q_c = .20$, the walk would begin with $A = 5$. This means that a priori there is about 70% assurance that the system is .80 reliable. If now ten actual successes and one actual failure were observed, a confidence level of about 80% has been indicated. In this manner, assumption of the initial values for A and B reduces the amount of tests required to demonstrate a reliability goal. However, reduced sampling increases the risk of a system demonstrating a reliability level when, in fact, it isn't that reliable. The Operating Characteristics curve is a graphic presentation of these risks.

Thus, poster size charts can be maintained on which test results are plotted in a step-by-step, random walk fashion as illustrated by the arrows in Figure 3. These charts can become powerful management tools for predicting reliability trends, for maintaining reliability "milestones", and in general, for monitoring overall reliability progress.

(The table of values used in the construction of Figures 2 through 3.5 is included in Table 2 of Appendix IX.)

9. Operating Characteristics

The operating characteristic of the procedure is defined as the probability of attaining the number of successes needed to demonstrate a specified reliability, $(1 - q_c)$, as a function of actual reliability, $(1 - q)$. Actually, the probability is treated as a function of the ratio $x = q/q_c$, where q is the actual unreliability and q_c is the specified unreliability.

If the reliability is high, $(x \ll 1)$, it is desirable to have a high probability of attaining the satisfactory number of successes. On the other hand, if the reliability is low $(x > 1)$, it is desirable to have a small probability of a satisfactory count.

Now consider the computation of the operating characteristics. Let A_i denote the number of successes, actual and constructed, required to demonstrate the specified reliability when i failures have occurred. For zero failures, $i = 0$, the probability of achieving A_0 successes is p^{A_0} or $(1 - q)^{A_0}$, since $p = (1 - q)$. The probability of attaining A_1 successes, experiencing no more than one failure, is written as:

$$(1-q)^{A_0} + A_0 q (1-q)^{A_1}$$

Letting $L(q)$ be the probability of attaining the required number of successes to demonstrate a given reliability, a more general expression would be:

$$L(q) = (1-q)^{A_0} + A_0 q (1-q)^{A_1} + \left[A_0(A_1 - A_0 + 1) + \frac{A_0(A_0-1)}{2}\right] q^2 (1-q)^{A_2} + \dots \tag{9}$$

The successive terms represent the probabilities of satisfying the required number of successes when failures, F, = 0, 1, 2, -------.

By letting $z_i = A_i\, q_c$ and $x = q/q_c$, and applying a limiting process as $q_c \rightarrow 0$, the expression (9) becomes:

$$L(q) = e^{-z_0 x} + z_0 x e^{-z_1 x} + \left[z_0(z_1 - z_0) + \frac{z_0^2}{2!}\right] x^2 e^{-z_2 x} + \ldots \tag{10}$$

The computation of $L(q)$ when considering larger number of failures becomes very laborious. In fact, the number of terms in z that form the coefficient for the exponential function increases by the power of two. This means that if one desires the probability of achieving the required number of successes having experienced no more than ten failures, the coefficient of the term in $L(q)$ associated with $x^{10} e^{-z_{10} x}$ would have 512 terms in z, instead of 2 as in the last term of (10). If $L(q)$ is extended to consider 11 failures, the additional coefficient would have 1024 terms in z. From this, it is clear that manual computing is impractical. (See Appendix VIII for details.)

The evaluation $L(q)$ up to 15 failures has been completed by use of an electronic digital computer. In order to evaluate the expression for 15 failures, 200 minutes of Univac 1103A machine computing time were utilized. (For a detailed discussion of the essential machine computing techniques, see Appendix X.)

Figures 4 through 5.43 are operating characteristics (O.C.) curves plotted from the results of the machine computation. From these charts, one can determine the risk of having a system demonstrate achievement of a reliability goal when, in fact, the system is below the specified reliability. The charts will also show the effect on this risk of assuming initial values for A and B.

For example, consider Figure 4, which displays O. C. curves for the 90% confidence level. Representative curves are plotted to illustrate how $L(q)$ varies with observed failures. The curves plotted are for three sets of assumed values for A and B and three failure conditions.

The risk of having a below par system accepted as having attained the reliability goal is the probability when $x > 1$. When $x > 1$, the actual q is greater than the specified q_c. Figure 4 shows that this risk varies from .13 to .33 for 0 through 15 failures for the unbiased condition.

The effect of the assumption of initial values for A and B is clearly indicated. As would be expected, the probabilities are increased as the bias is increased. Figures 5.11 through 5.43 present a range of curves for 70%, 80%, 90%, and 95% confidence levels. Each chart displays three bias conditions.

10. Components

One should next extend this model to the counting of successes and failures of components for a system. Suppose that a system is to contain n independent components, no one of which is known a priori to be definitely more reliable than the others. Therefore, we shall assume the same a priori probability distribution for the failure probabilities Q_1, Q_2, ----, Q_n, of the n components. This distribution is of the form of (1) with the parameters A and B to be specified subsequently. The choice of these parameters, like the earlier choice (8), will be dictated by an a priori distribution, i.e., a subjective opinion of the system as a whole.

Let q_i denote the probability of failure of the i^{th} component, $i = 1, 2, ----, n$, and q denote the system failure probability, then:

$$q = 1 - \prod_{i=1}^{n} (1 - q_i) \tag{11}$$

Since the individual component failure probabilities must be very small in order to achieve high reliability, (11) may be approximated by:

$$q = q_1 + q_2 + q_3 + \ldots\ldots\ldots q_n \tag{12}$$

Also, it is pointed out that formula (12) holds in cases where it is realistic to assume that system failure results from one and only one component failure. In this case, no assumption is made regarding independence of components.

From (2) and (12) it follows that the a priori distribution of system unreliability, Q, has a mean:

$$(13) \qquad \mu_q \cong \sum_{i=1}^{n} \mu_{q_i} \cong \frac{n(B+1)}{A}$$

and a standard deviation:

$$(14) \qquad \sigma_q \cong \sqrt{\sum_{i=1}^{n} \sigma_{q_i}^2} \cong \frac{\sqrt{n(B+1)}}{A}$$

Assume, as was done in deriving (8), that the a priori distribution of Q has a mean at q_c and a standard deviation also equal to q_c. Then, from (13) and (14), it is seen that one must choose:

$$(15) \qquad \begin{aligned} A &= \frac{1}{q_c} \\ B &= -\frac{n-1}{n} \end{aligned}$$

It is worth noticing that this means, according to (2), the individual component failure probabilities q_i have an a priori distribution with mean of the order $\mu q_i = \frac{q_c}{n}$. Accordingly, the standard deviation $\sigma q_i = \frac{q_c}{\sqrt{n}}$, though small, is not of the same order of smallness as μq_i. This a priori assumption about components, which weights the extremes (q close to 0 and q close to 1) more heavily than the distribution indicated in (8), is necessary in order that the a priori distribution of the overall system shall not be concentrated closely around its assumed mean, q_c. Such a concentration would certainly be unjustified a priori.

Now that the a priori component distribution parameters A and B have been chosen according to (15), the a posteriori distributions are found as before. Thus, if tests of the i^{th} component have yielded S_i actual successes and F_i actual failures, the a posteriori distribution Q_i is characterized by:

$$(16) \qquad \begin{cases} A_i = 1/q_c + S_i \\ B_i = -\frac{n-1}{n} + F_i \end{cases}$$

thus yielding a mean

$$(17) \qquad \mu_{q_i} \cong \frac{1/n + F_i}{1/q_c + S_i}$$

and a standard deviation

$$(18) \qquad \sigma_{q_i} \cong \frac{\sqrt{1/n + F_i}}{1/q_c + S_i}$$

The a posteriori distribution of system failure probability Q then has a mean:

$$(19) \qquad \mu_q \cong \sum_{i=1}^{n} \frac{1/n + F_i}{1/q_c + S_i}$$

and a standard deviation

$$(20) \qquad \sigma_q \cong \sqrt{\sum_{i=1}^{n} \frac{1/n + F_i}{(1/q_c + S_i)^2}}$$

Making a change in variables and considering the limiting case as in deriving (7), one may write:

$$\alpha(z_c) = \int_0^{z_c} \frac{e^{-z} z^{B}}{\Gamma(B+1)} dz \tag{21}$$

where the B is the single parameter of the system and is associated with failures in the system. B_i is the number of failures (actual and constructed) in the i^{th} component. The relationship between B and B_i is the following:

$$\begin{aligned} \sum_{i=1}^{n}(B_i+1) &= B+1 \\ \therefore B &= n-1+\sum_{i=1}^{n} B_i \end{aligned} \tag{22}$$

Substituting the a posteriori B_i from (16) leads to:

$$\begin{aligned} \sum_{i=1}^{n} B_i &= \sum_{i=1}^{n} F_i-(n-1) \\ B &= \left[\sum_{i=1}^{n} F_i-(n-1)\right]+(n-1) \\ \therefore B &= \sum_{i=1}^{n} F_i \end{aligned} \tag{23}$$

This is a very important relationship because it permits the integration of component test results to demonstrate system reliability. Therefore, the proposed method is readily adaptable to a developmental program, where each component designer may be conducting separate tests. The system reliability can be computed at any time during this separate testing phase. If all components have been subjected to an equal amount of tests, the determination of the parameter B is simply the sum of all failures experienced by each

of the components. It is unlikely, however, that all components will have been tested equally. It is then possible to compute a conservative lower bound to the reliability by disregarding, for the moment, the number of successes over the minimum number recorded for any one component. Similarly, an optimistic, upper bound can be computed by using the maximum number of successes experienced by any one component as the number of system successes.

Example: Consider a complete system of 3 components which have been undergoing separate testing programs. The following results have been observed:

Component A : 22 successes, 1 failure

Component B : 26 successes, 1 failure

Component C : 24 successes, 0 failure

Referring to Figure 2, the upper and lower bounds to a system reliability are computed as follows:

Upper (26 successes, 2 failures) : $q_c = .20$

Lower (22 successes, 2 failures) : $q_c = .25$

Therefore, one may state with 90% confidence that the system reliability, at this stage of the development, is in the interval .75 to .80.

The preceding discussion has assumed that the development of all components of the system progresses simultaneously. In practice, however, it may be found that while the development of most components is proceeding in step, several components may be far behind. It is not necessary to wait for these "stragglers" to reach the same state of development before indicating system reliability.

In the example, consider the case where component A has not been tested and the test results of component B and C are as indicated. Again from Figure 3, the indication is that system reliability, based only on components B and C, is about .85. From this, management can reason that if the development of component A is as satisfactory as that for components B and C, the system reliability will be in an interval around .85. At some subsequent time, the results of tests with component A can easily be combined with those of B and C, to yield a modified system reliability figure.

11. Expected Number of Tests

For planning purposes, it is desirable to determine the number of tests one should expect to conduct in order to demonstrate attainment of the reliability goal.

Since this method is "one-sided", in that there is no "reject" criteria, the expected number of tests required for reliability demonstration increases as more failures occur. Therefore, in order for an expected number of tests to be calculated, the number of failures to be considered must be specified. For example, one must ask, "What is the expected number of tests one must conduct in order to demonstrate achievement of the reliability goal if no more than 'k' failures are observed?"

Because one assumption throughout has been that the number of successes is much greater than the number of failures, ($A \gg B$), the number of successes is essentially the number of tests. Letting N equal the expected number of tests, one may write:

$$(24) \qquad N = A_0 P_0 + A_1 P_1 + \ldots + A_{k-1} P_{k-1} + A_k \left[1 - (P_0 + P_1 + \ldots + P_{k-1})\right]$$

which is the number of successes required to reach the "index" curve for a specified q_c multiplied by the probability of achieving this number of successes. Recalling that $z_1 = A_1 q_c$, it is convenient to rewrite (24) as:

$$(25) \qquad q_c N = z_0 P_0 + z_1 P_1 + \ldots + z_{k-1} P_{k-1} + \left[1 - (P_0 + P_1 + \ldots + P_{k-1})\right]$$

In (24) and (26), the probabilities P_0, P_1, P_2, ... P_k are the values of the term of the L(q) expression that is associated with the number of failures denoted by the subscript.

For any given confidence and assumed value for A, which determines the value of the z_i, equation (25) can be plotted versus the ratio of actual to specified unreliability, $x = q/q_c$. Figure 6 is just such a graph. The figure is for the case where A = 0 and a 90% confidence level.

Figures 6.1 and 6.2 are for the same confidence but illustrate the effect of two different assumed values of A on the expected number, N.

To show how N varies as the x ratio, the assumed value of A, and the number of failures vary, the following table has been prepared from the data disolayed by Figures 6 through 6.2.

EXPECTED NUMBER OF TESTS FOR q_c = .20, 90% CONFIDENCE

	$x = q/q_c$								
	x = .5			x = 1.0			x - 2.0		
F \ A	0	$1/2q_c$	$1/q_c$	0	$1/2q_c$	$1/q_c$	0	$1/2q_c$	$1/q_c$
5	27	21	16	41	35	30	46	43	38
10	31	25	19	64	55	47	76	73	67
15	33	27	20	84	76	62	106	106	94

CURVES FOR DETERMINING UNRELIABILITY q WITH z VALUES FOR SEVERAL CONFIDENCE LEVELS

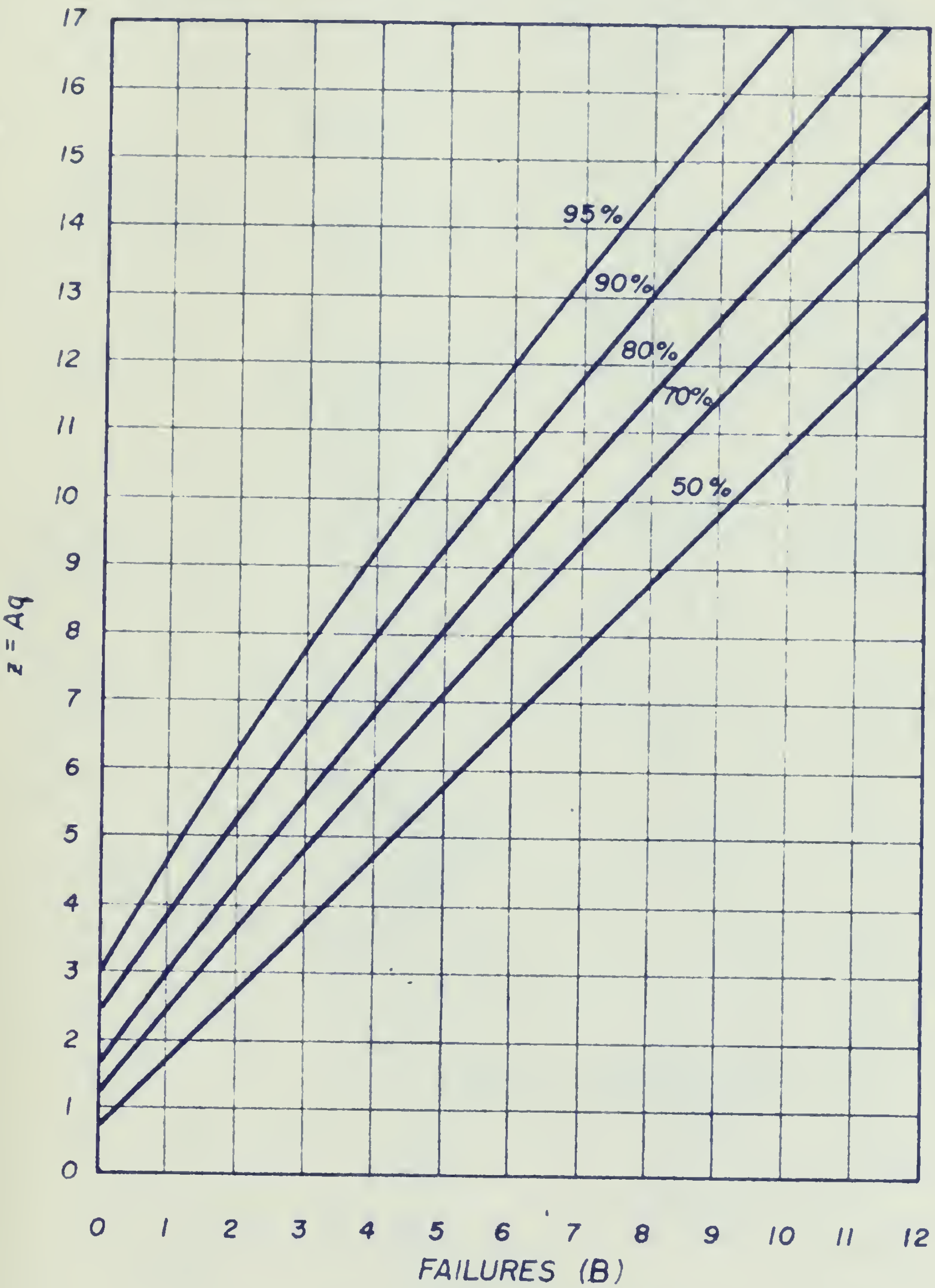


FIGURE 1

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED FOR A SPECIFIED CONFIDENCE

(EXPANDED SCALE)

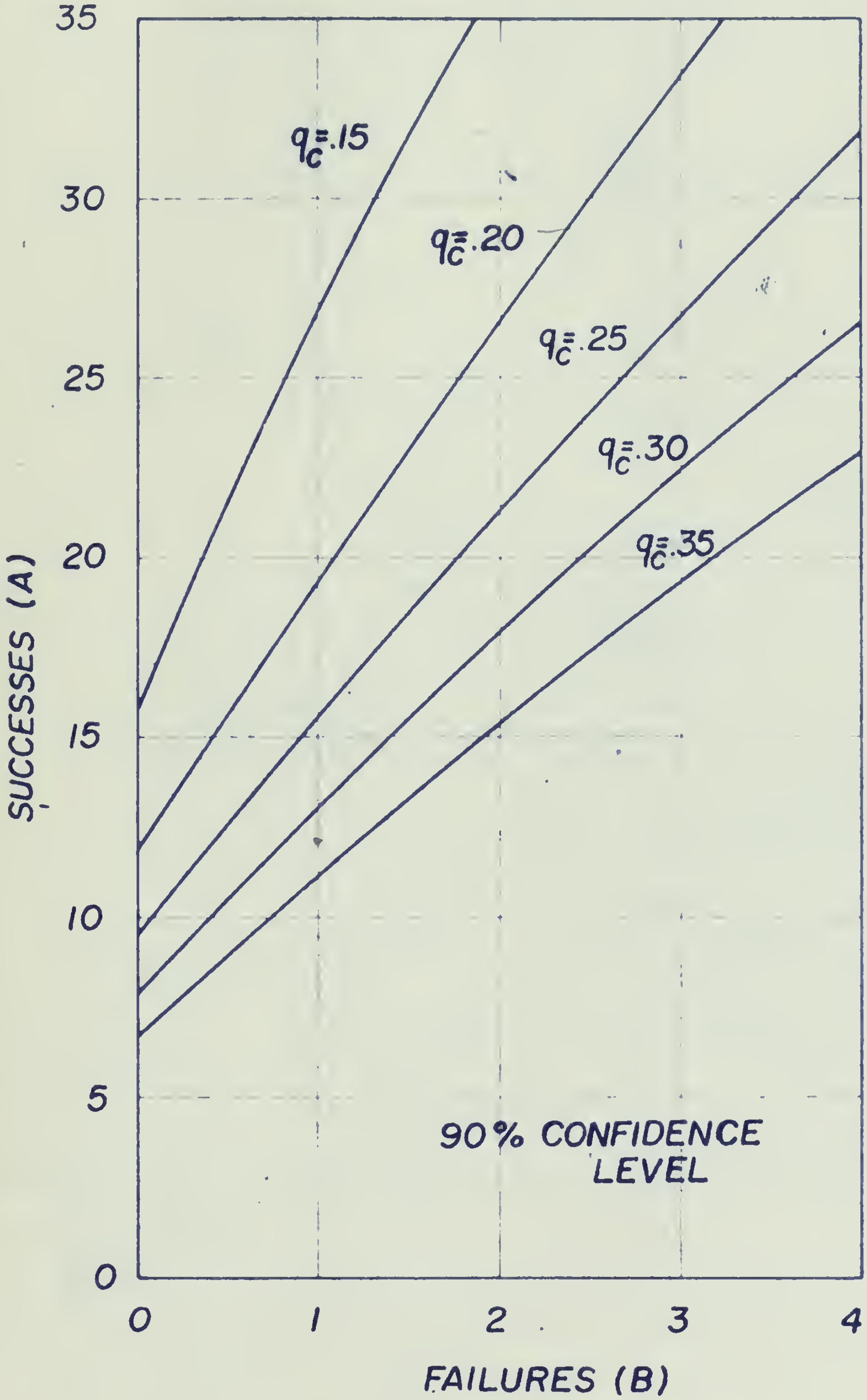


FIGURE 2

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED CONFIDENCE FOR SEVERAL UNRELIABILITY LEVELS q_c. (A PRIORI HYPOTHESIS: A=B=0)

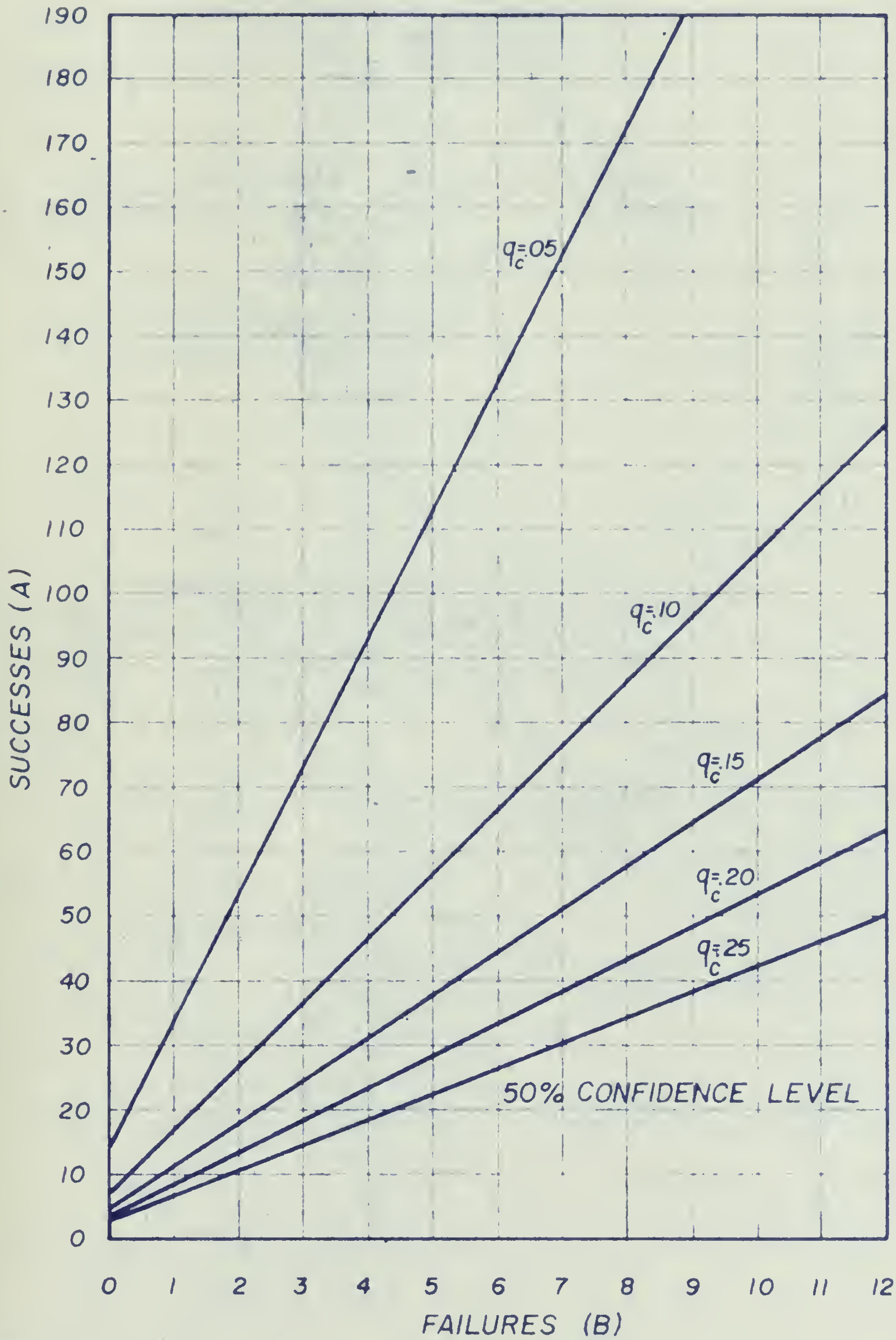


FIGURE 2.1

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED CONFIDENCE FOR SEVERAL UNRELIABILITY LEVELS q_c. (A PRIORI HYPOTHESIS: A=B=0)

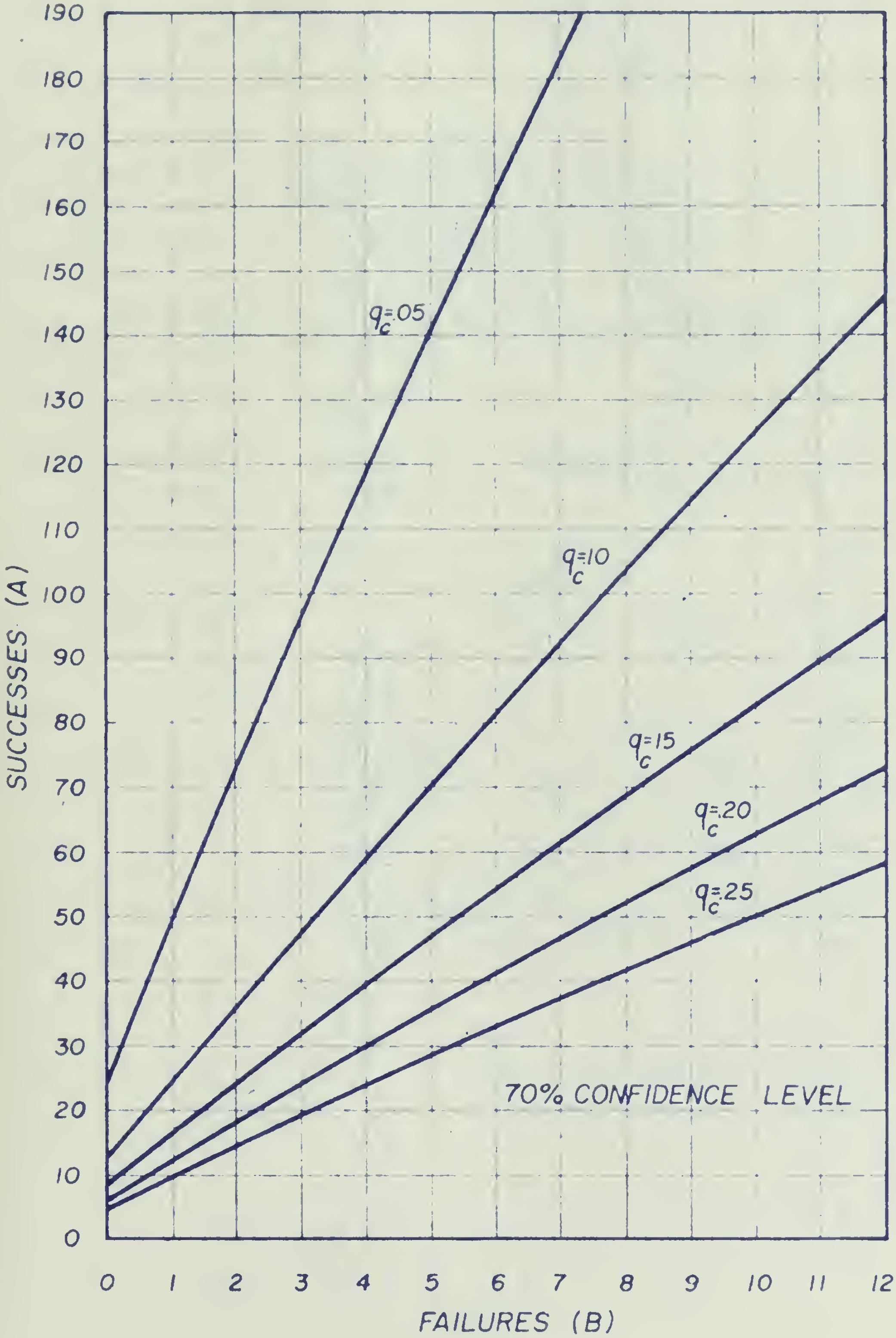


FIGURE 2.2

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED CONFIDENCE FOR SEVERAL UNRELIABILITY LEVELS q_c. (A PRIORI HYPOTHESIS: A=B=0)

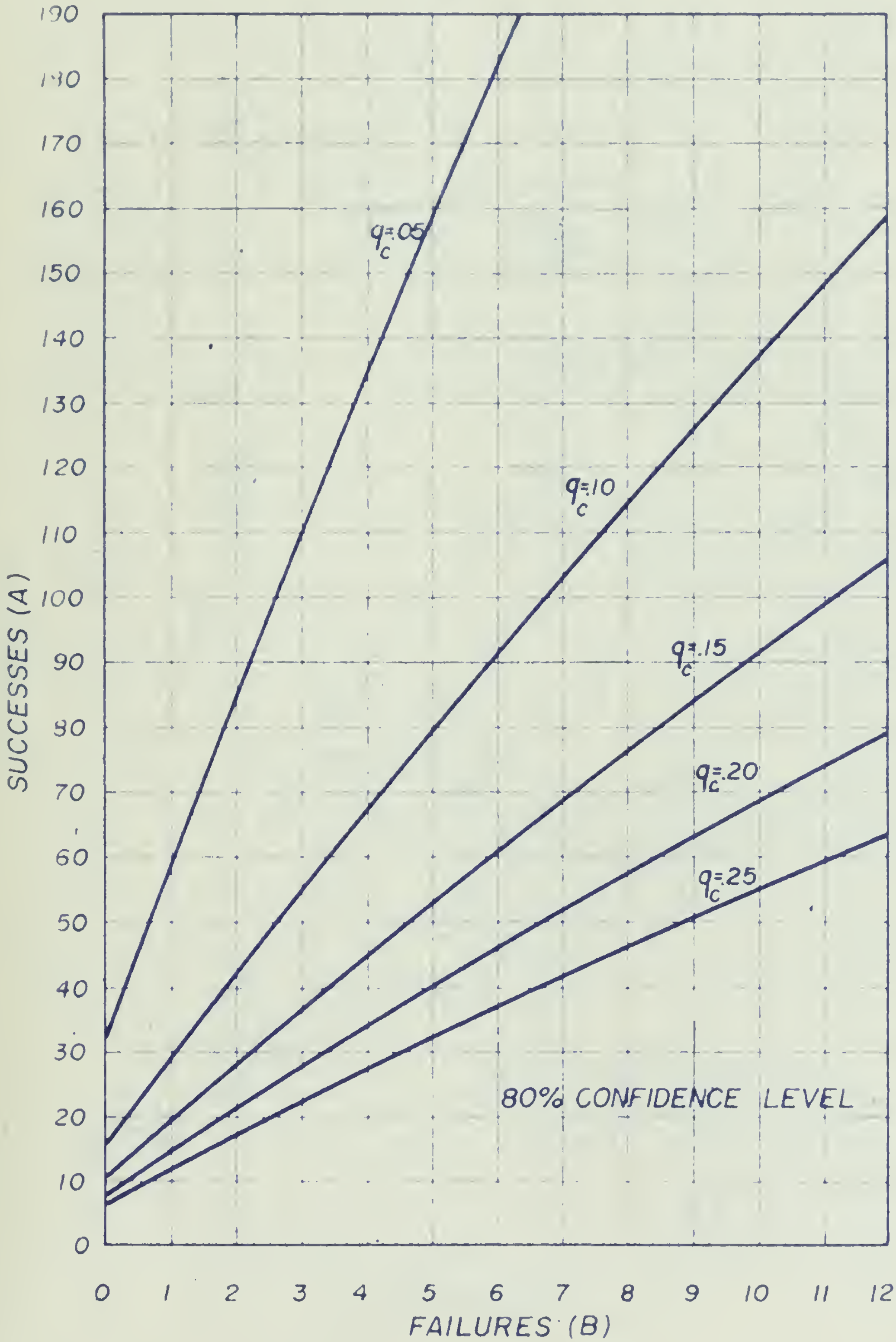


FIGURE 2.3

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED CONFIDENCE FOR SEVERAL UN-RELIABILITY LEVELS q_c. (A PRIORI HYPOTHESIS: A=0, B=0)

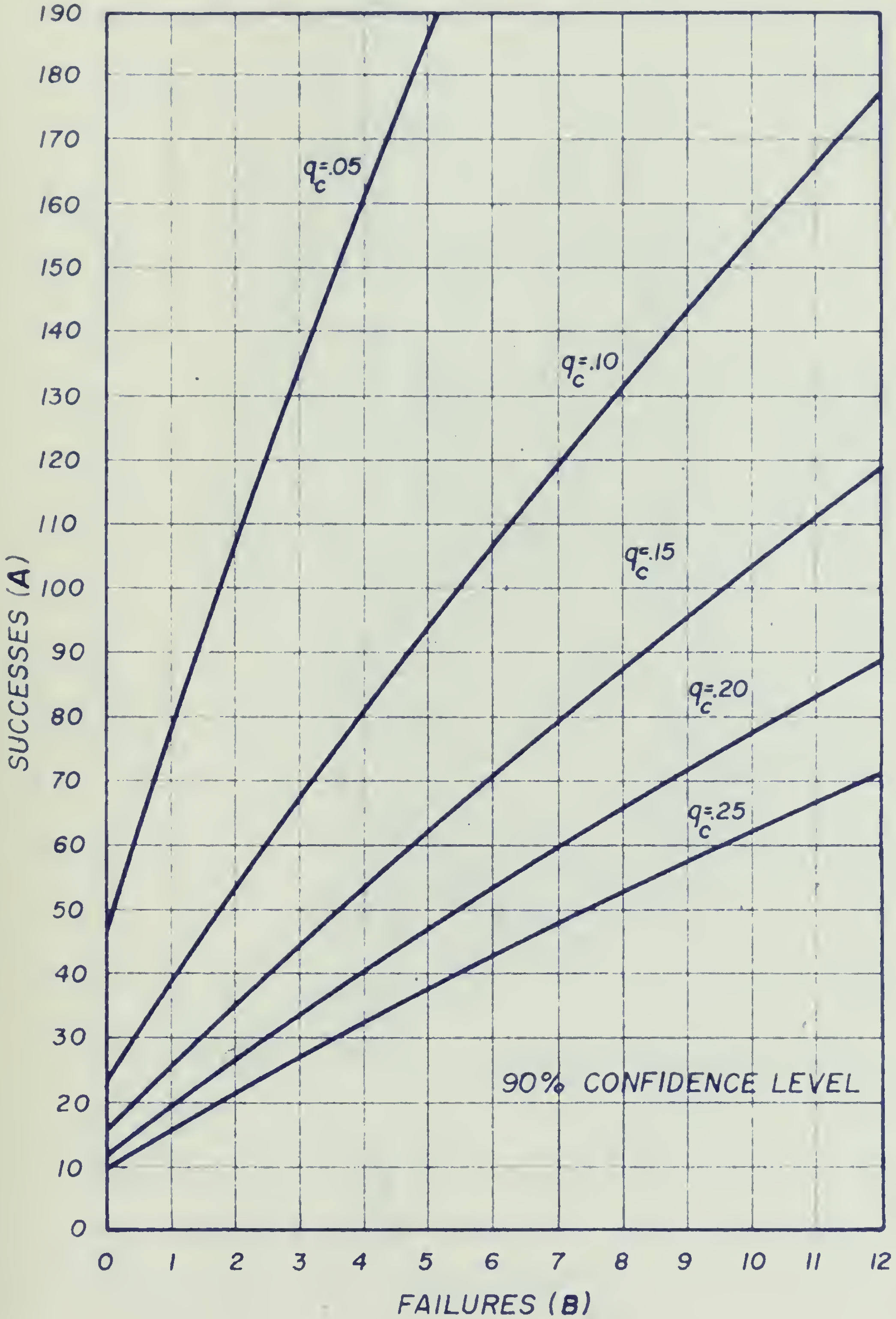


FIGURE 2.4

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED CONFIDENCE FOR SEVERAL UNRELIABILITY LEVELS q_c. (A PRIORI HYPOTHESIS; A=B=0)

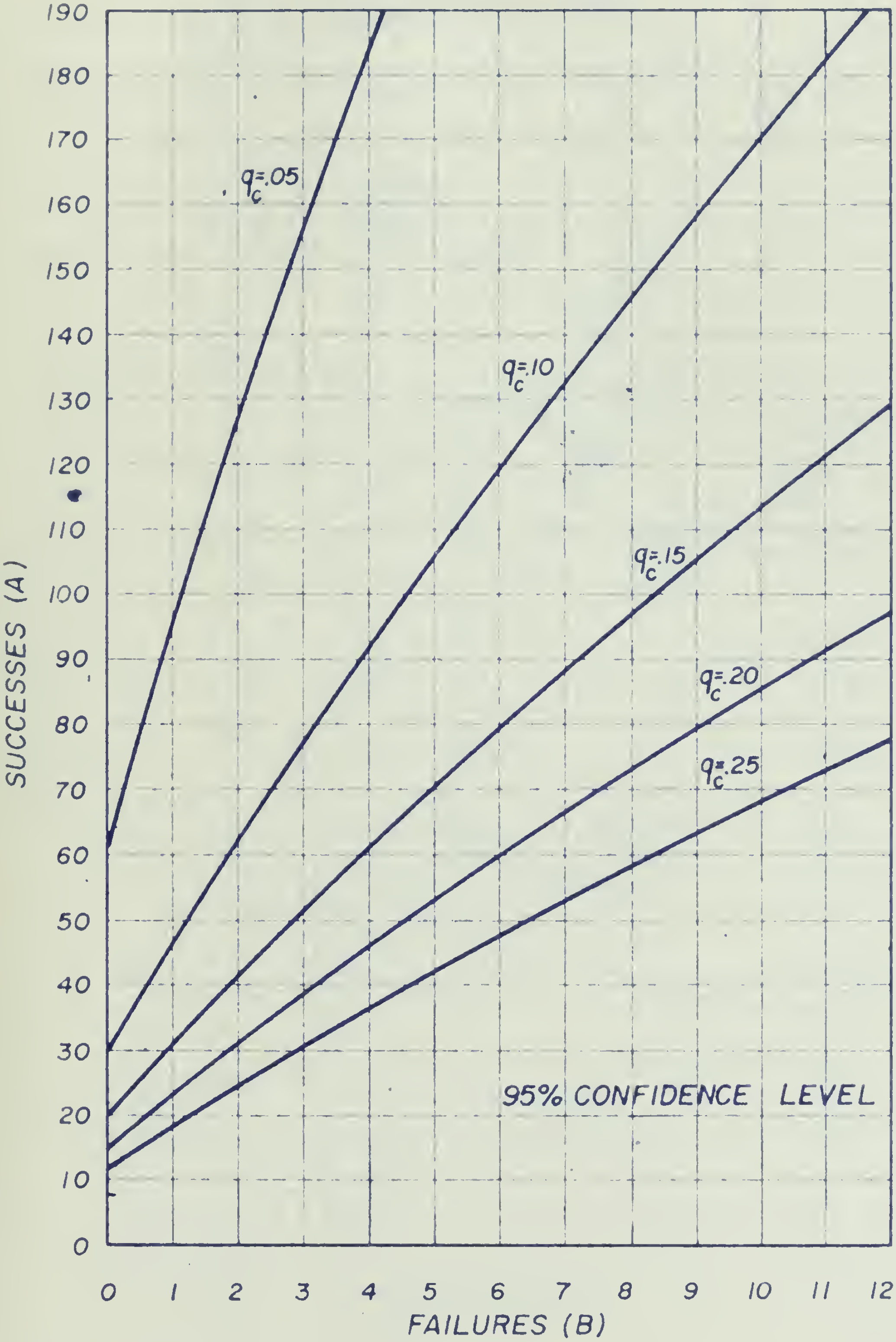


FIGURE 2.5

THE DEMONSTRATION WALK
(EXPANDED SCALE)

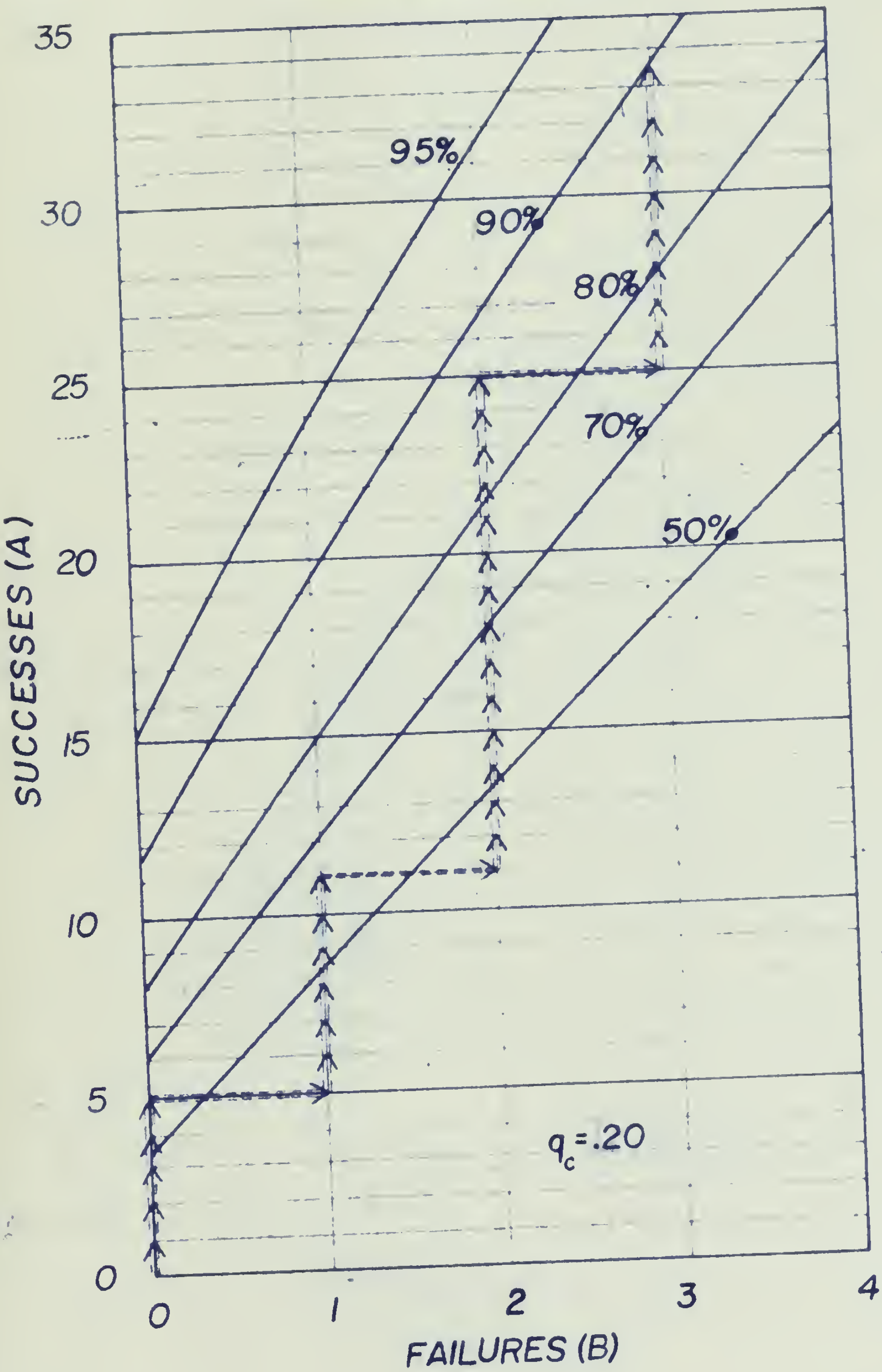


FIGURE 3

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED UNRELIABILITY q_c FOR SEVERAL CONFIDENCE LEVELS (A PRIORI HYPOTHESIS: A = B = 0)

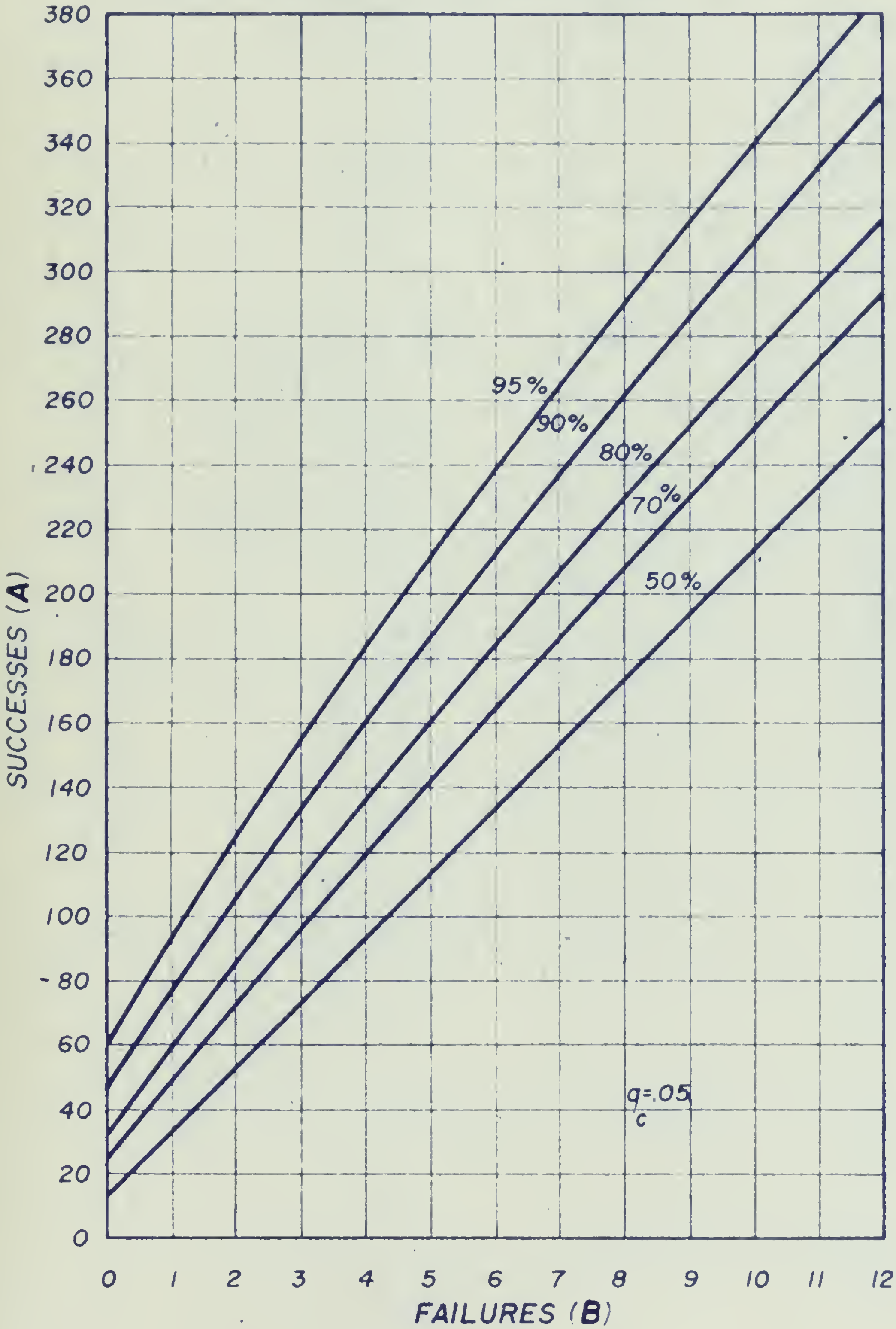


FIGURE 3.1

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED UNRELIABILITY q_c FOR SEVERAL CONFIDENCE LEVELS (A PRIORI HYPOTHESIS: A=B=0)

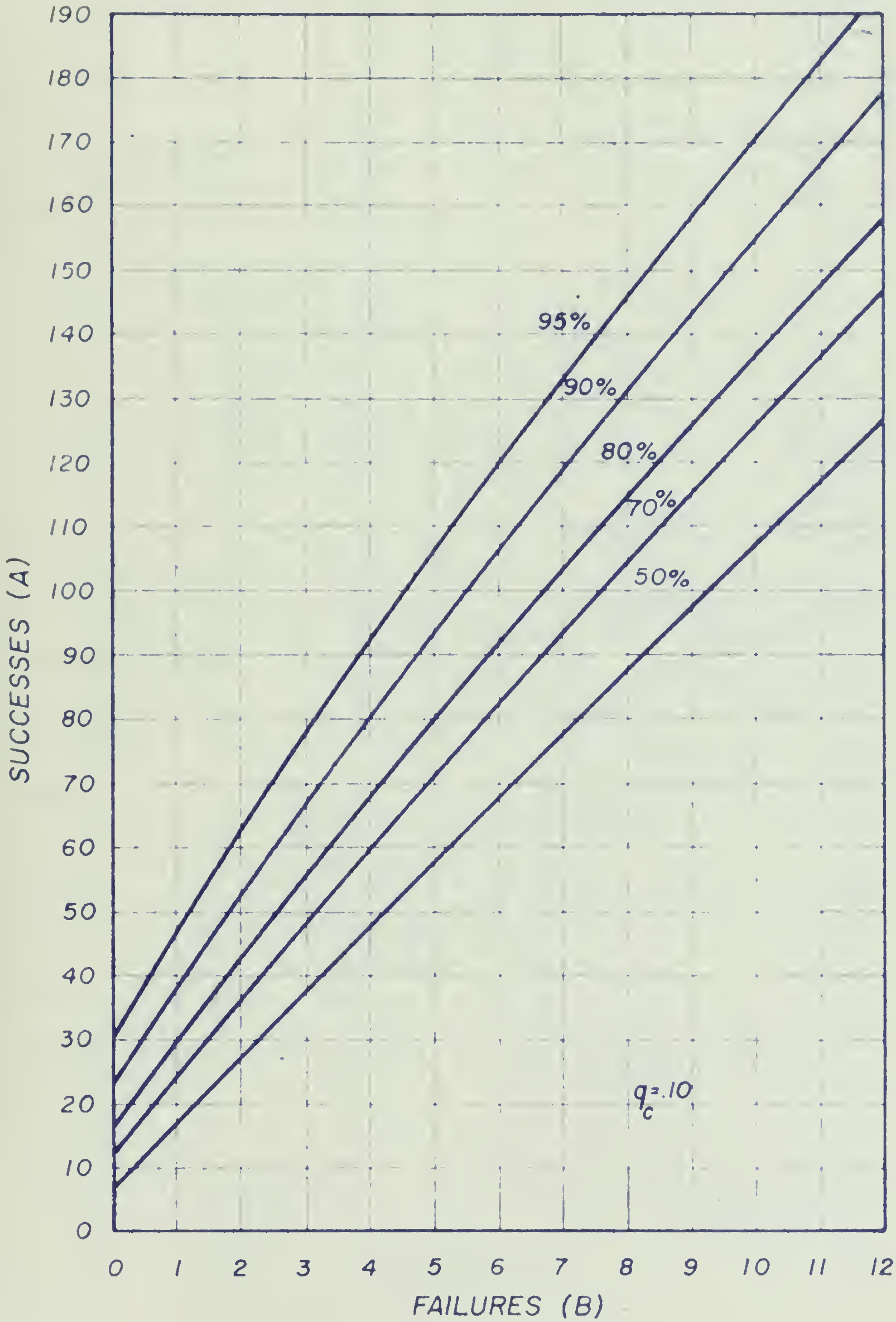


FIGURE 3.2

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED UNRELIABILITY q_c FOR SEVERAL CONFIDENCE LEVELS (A PRIORI HYPOTHESIS: A = B = 0)

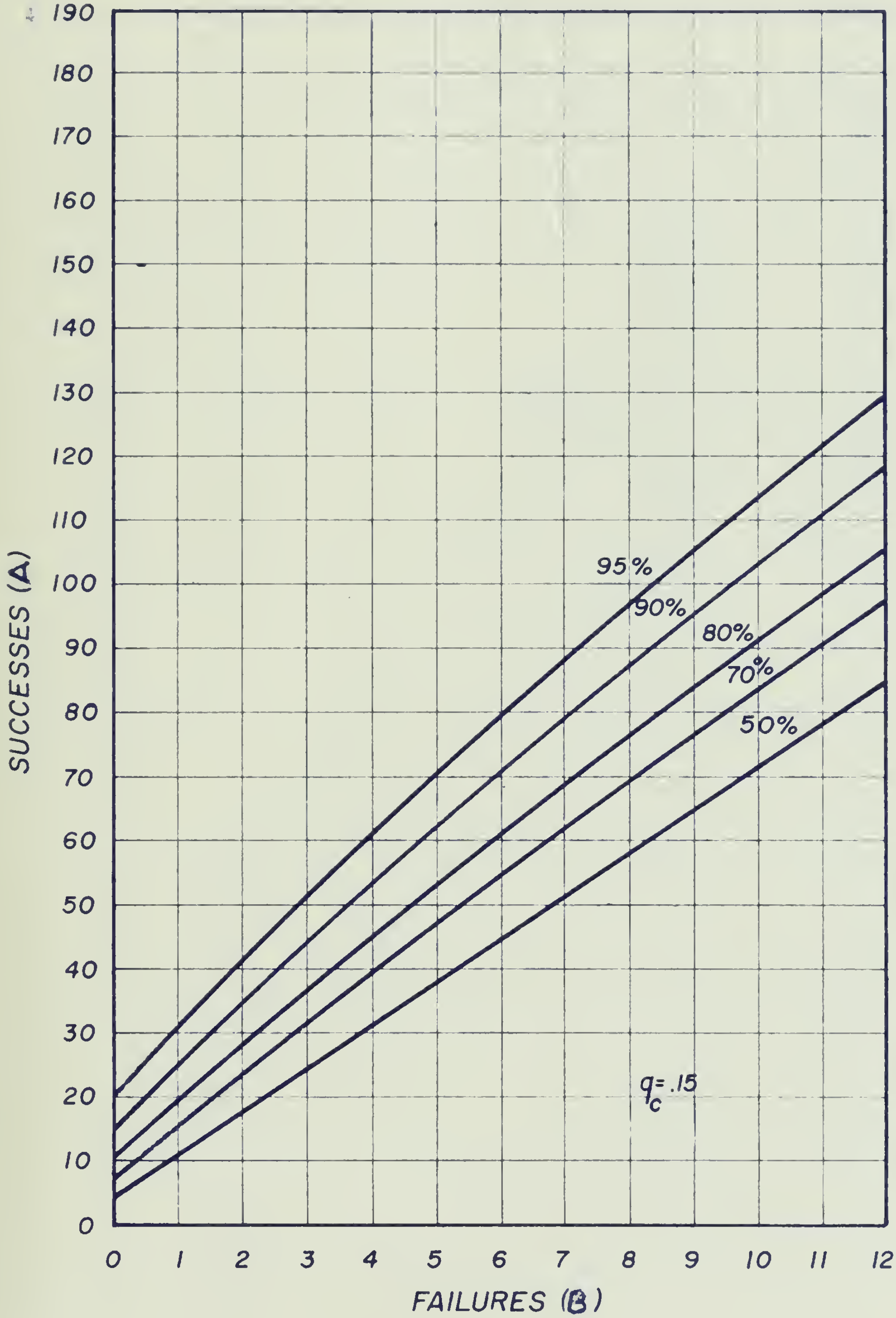


FIGURE 3.3

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED UNRELIABILITY q_c FOR SEVERAL CONFIDENCE LEVELS (A PRIORI HYPOTHESIS: A=B=0)

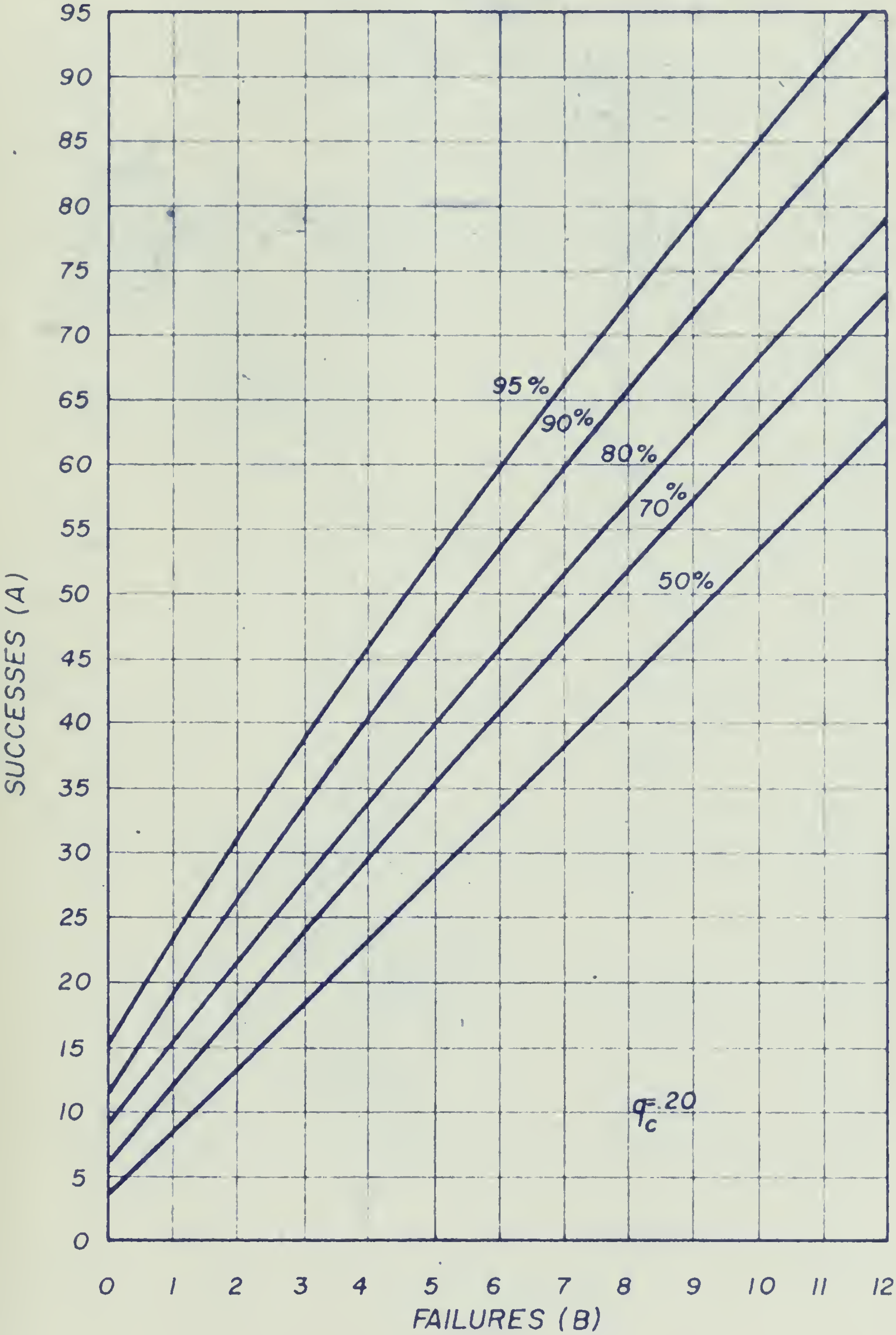


FIGURE **3.4**

CURVES FOR DETERMINING NUMBER OF SUCCESSES REQUIRED AT A SPECIFIED UNRELIABILITY q_c FOR SEVERAL CONFIDENCE LEVELS (A PRIORI HYPOTHESIS: A=B=O)

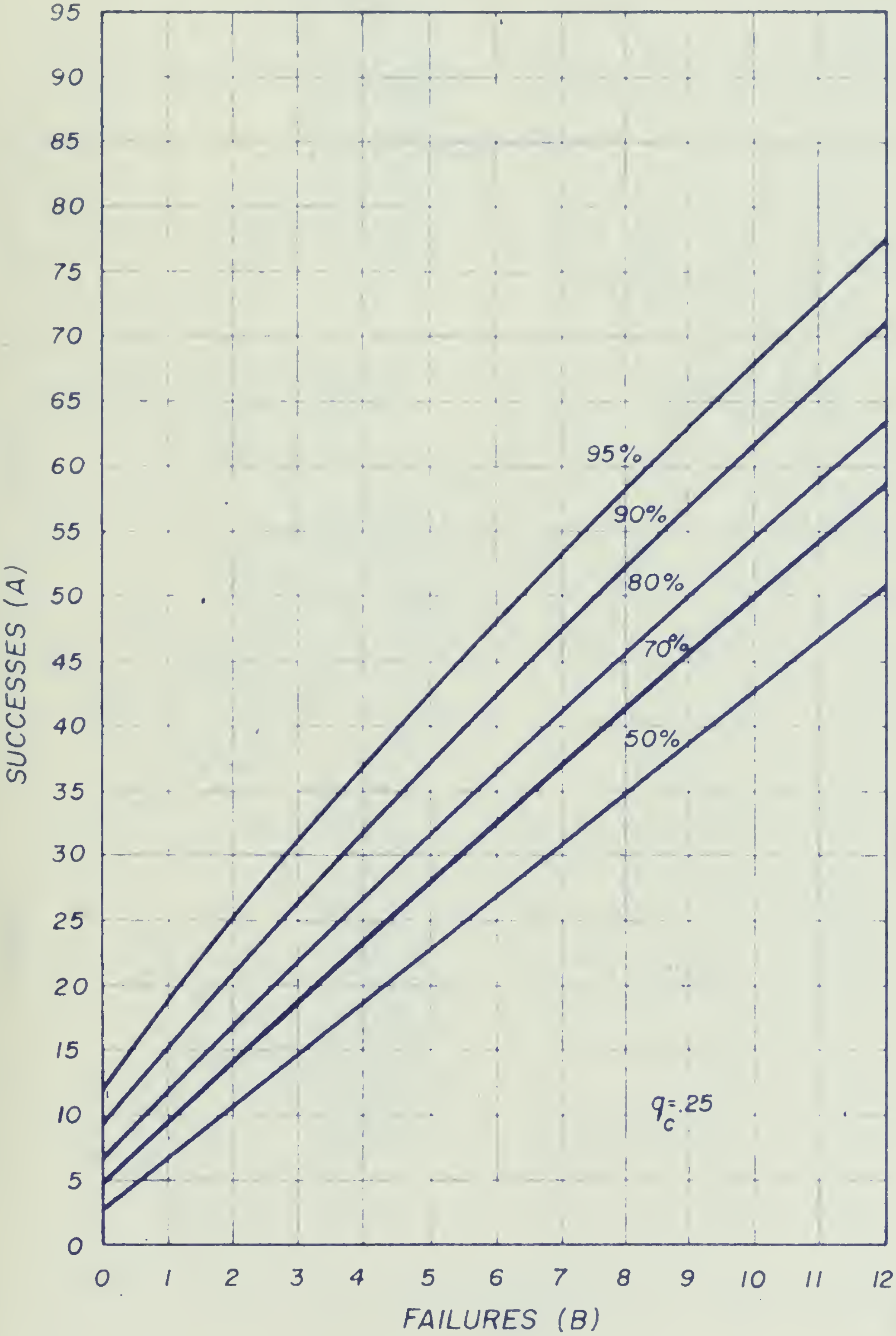
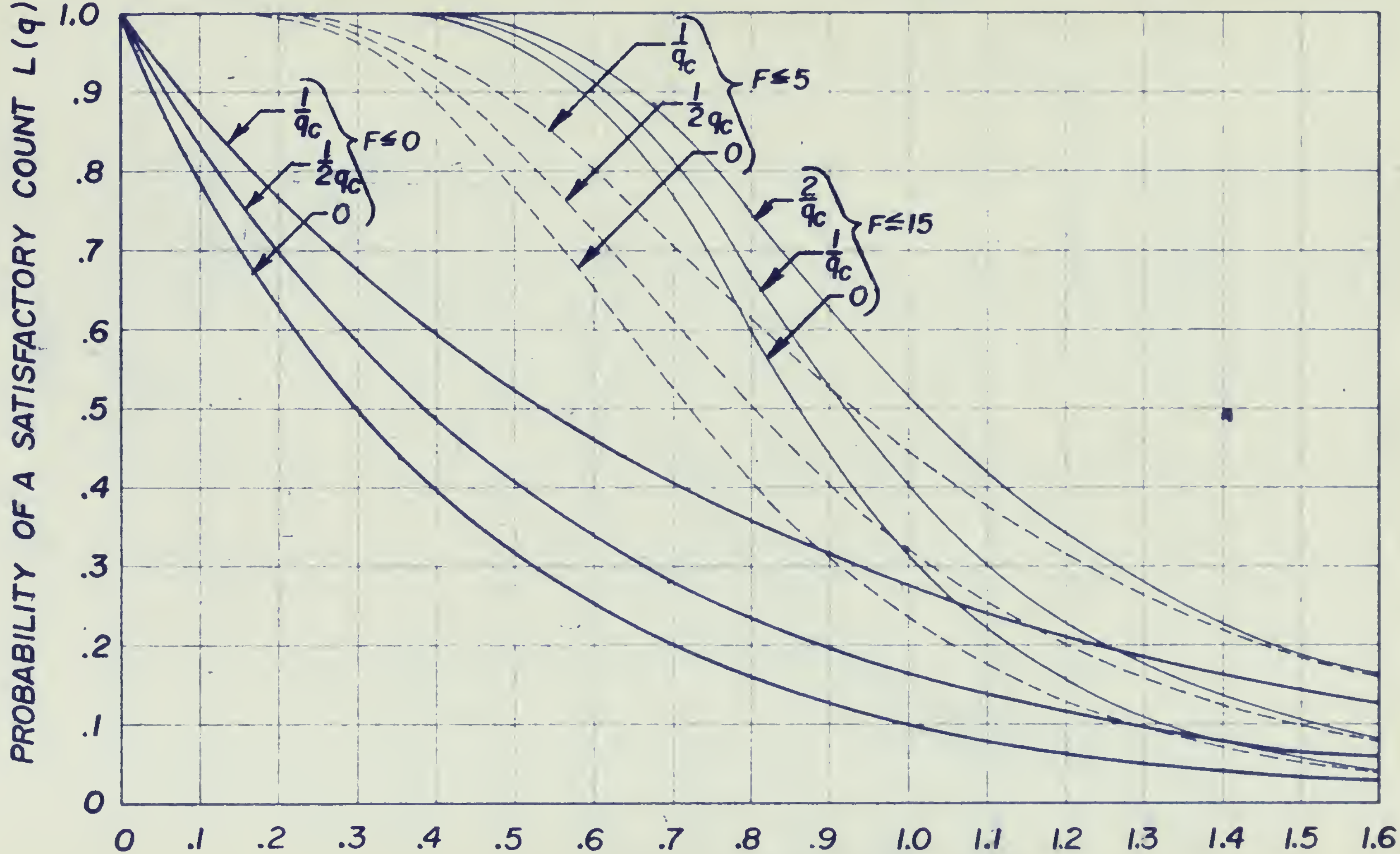


FIGURE 3.5

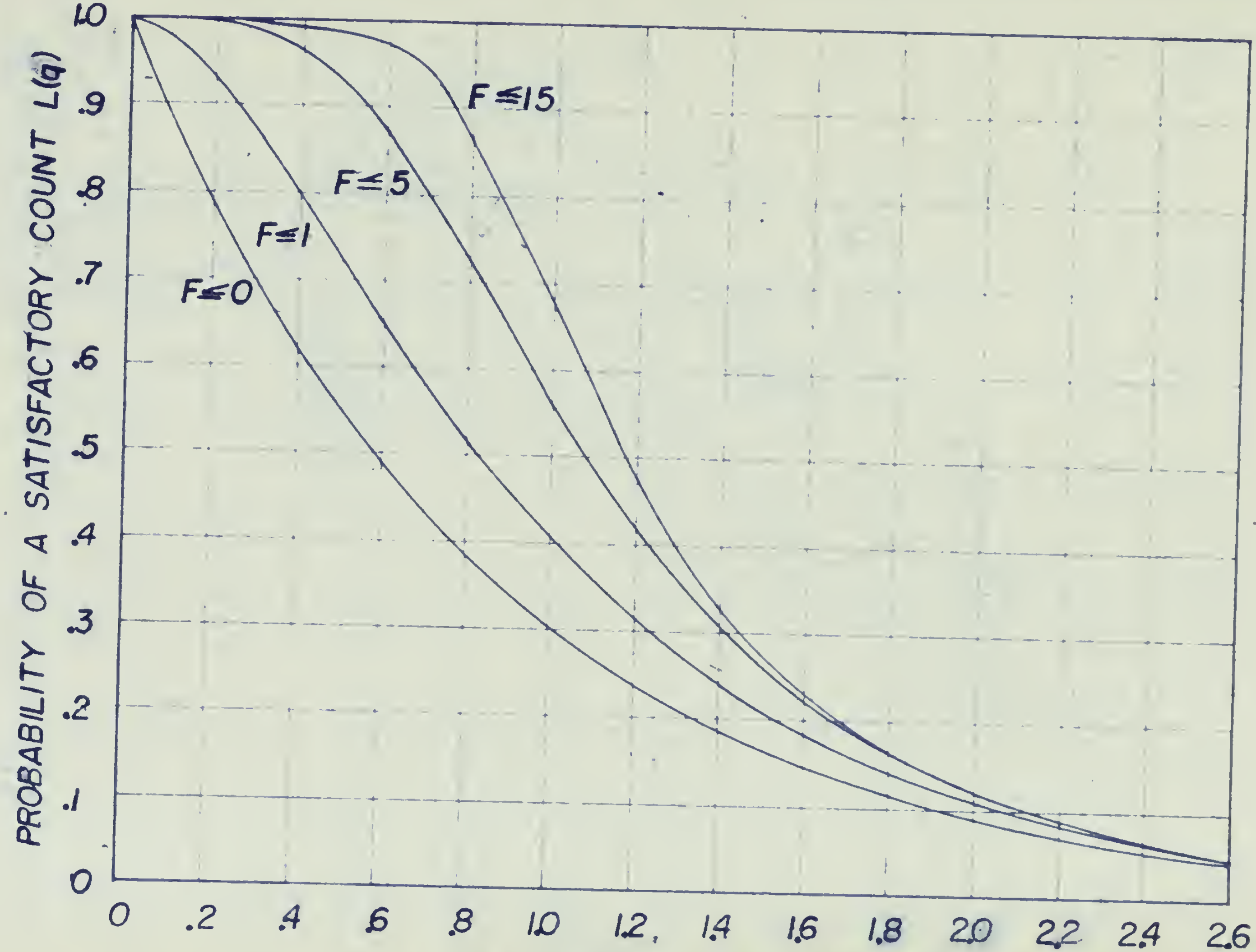
COMPARISON OF THE OPERATING CHARACTERISTIC CURVES
FOR VARIOUS A PRIORI HYPOTHESES (90% CONFIDENCE)



$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$

FIGURE 4

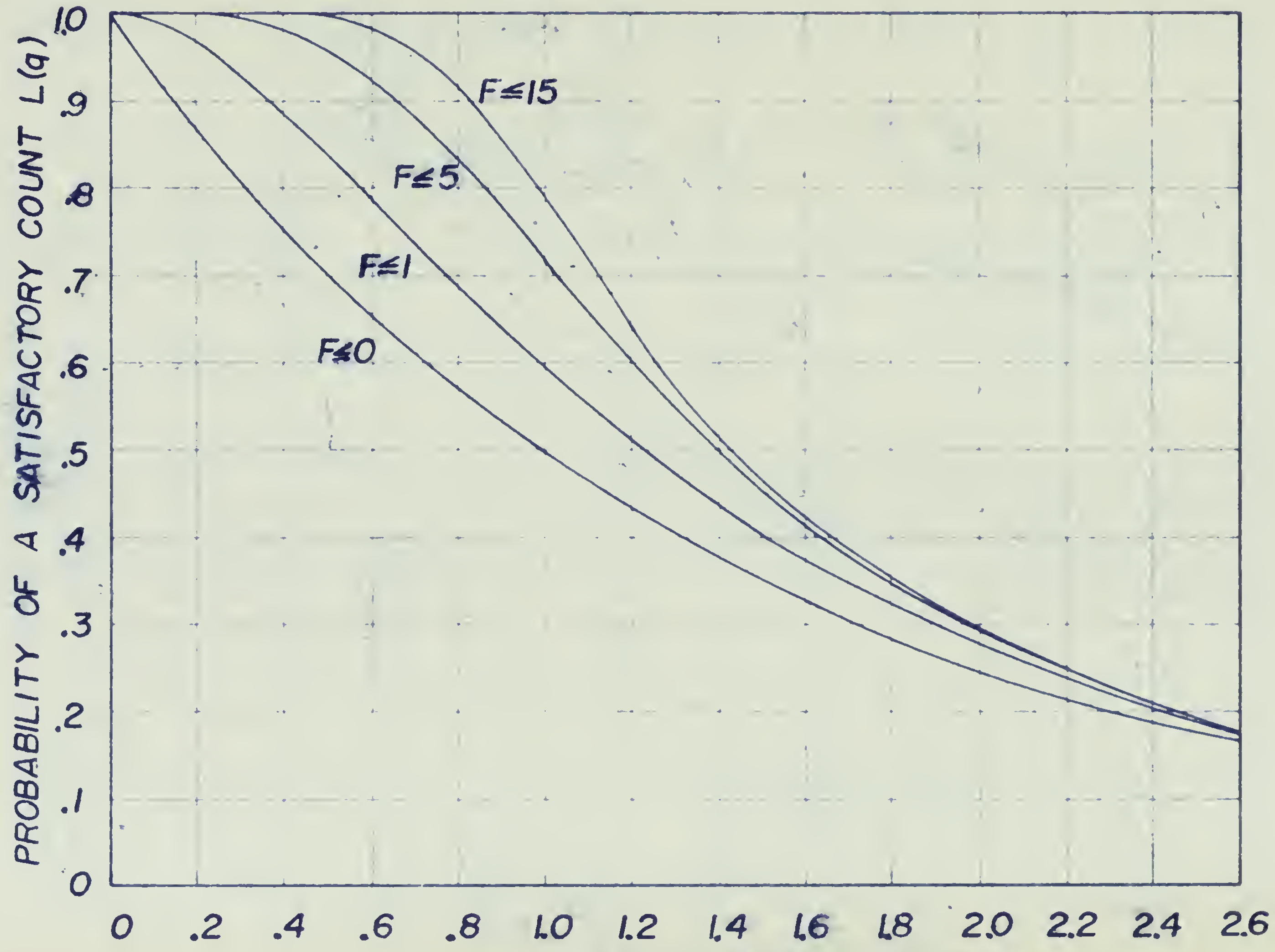
OPERATING CHARACTERISTIC CURVES AT 70%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: A=B=0)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.11

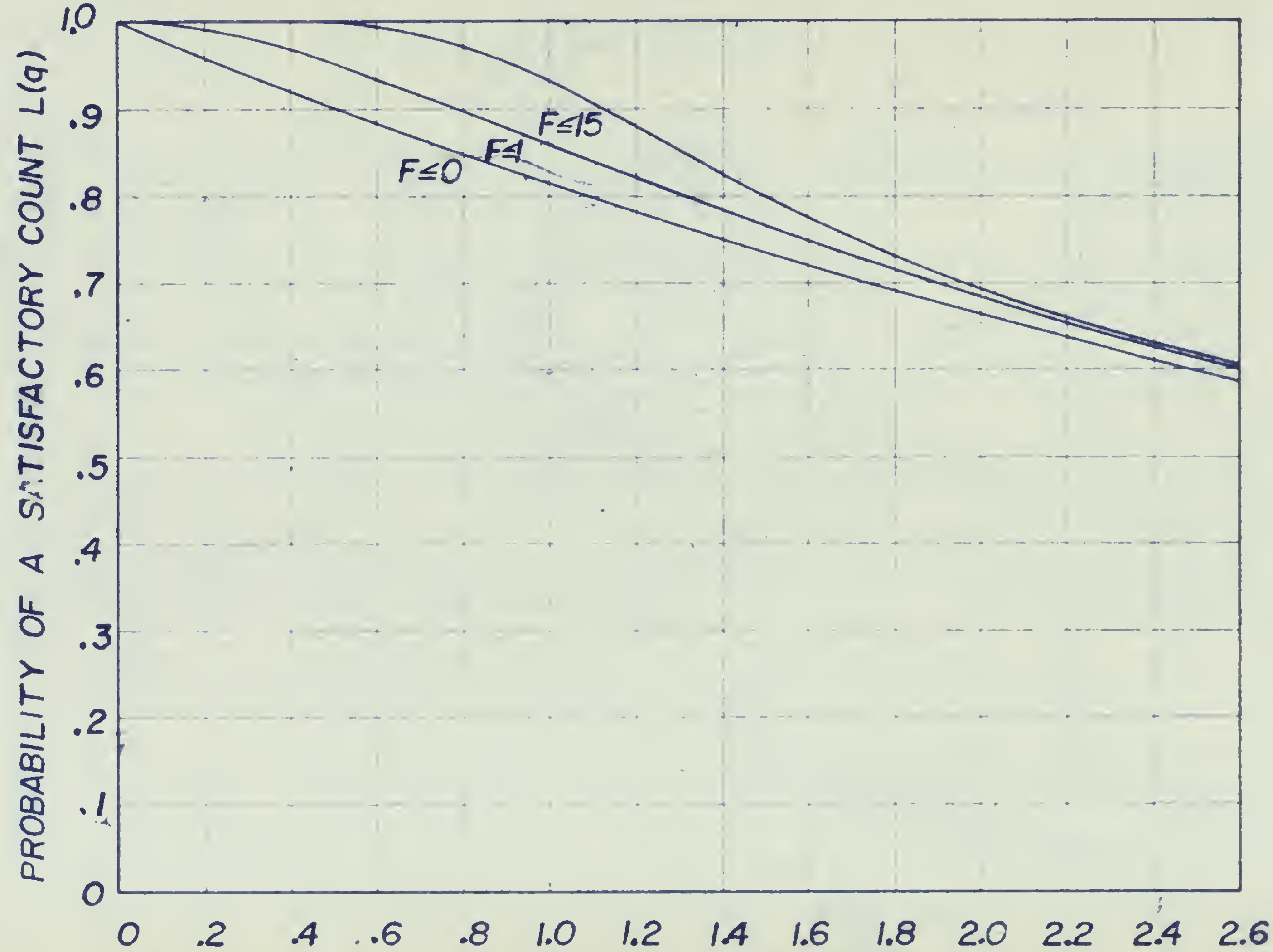
OPERATING CHARACTERISTIC CURVES AT 70% CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A=\frac{1}{2q_c}, B=0$)



$$X=\frac{q}{q_c}=\frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.12

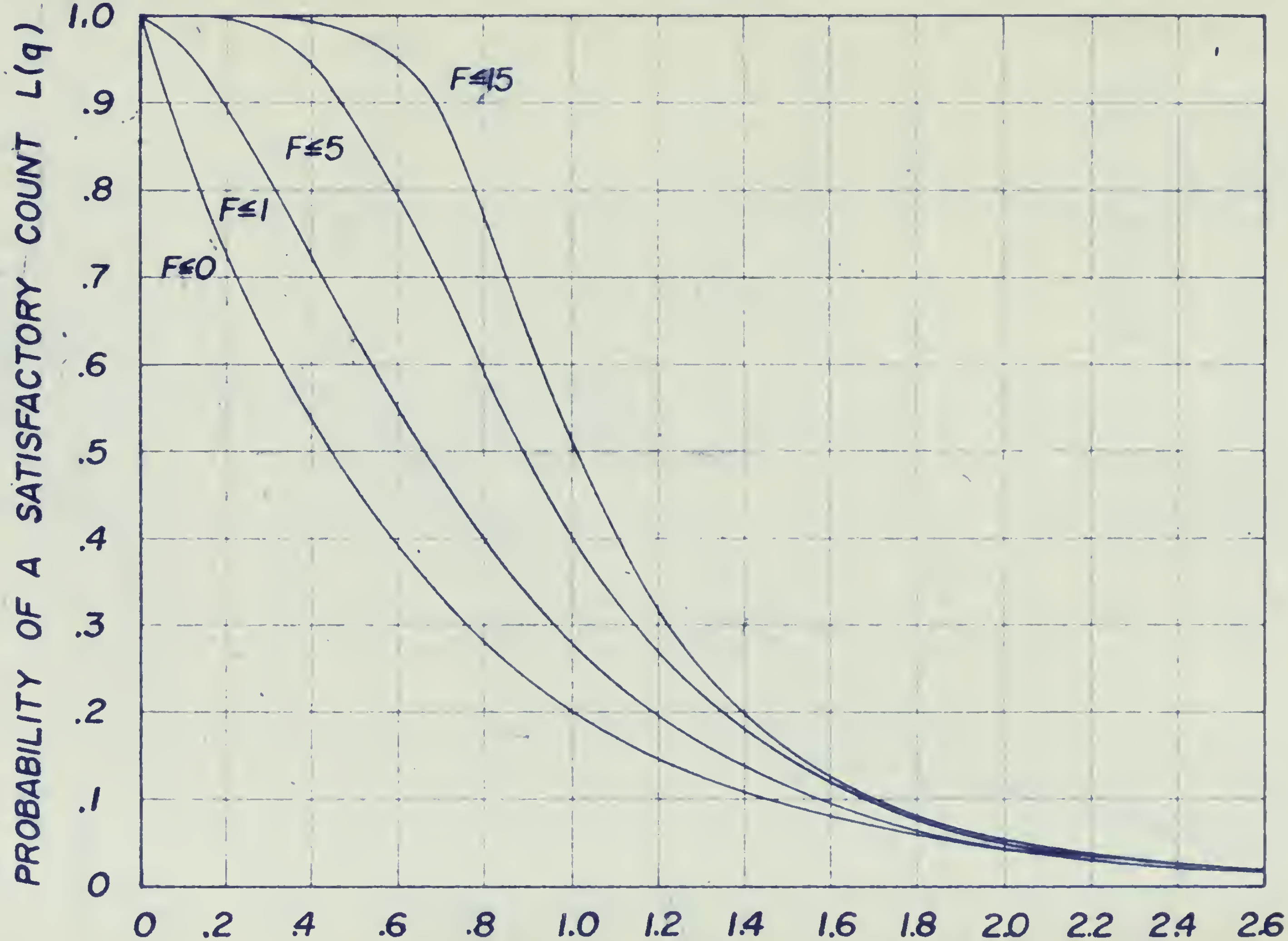
OPERATING CHARACTERISTIC CURVES AT 70% CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS $A=\frac{1}{q_c}, B=0$)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.13

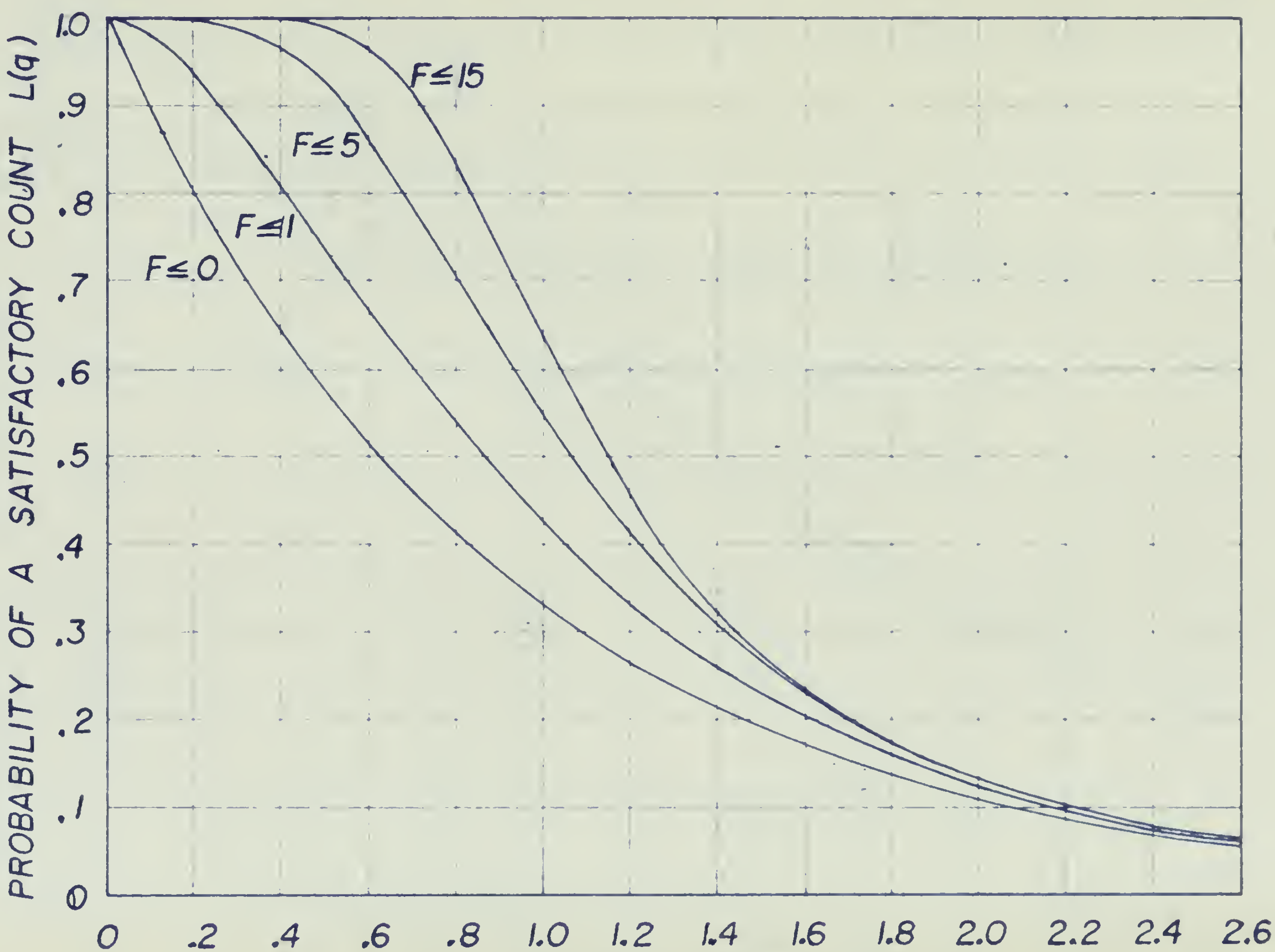
OPERATING CHARACTERISTIC CURVES AT 80%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: A=B=O)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.21

OPERATING CHARACTERISTIC CURVES AT 80 %
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A=\frac{1}{2q_c}, B=0$)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.22

OPERATING CHARACTERISTIC CURVES AT 80%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A = \frac{1}{q_c}, B = 0$)

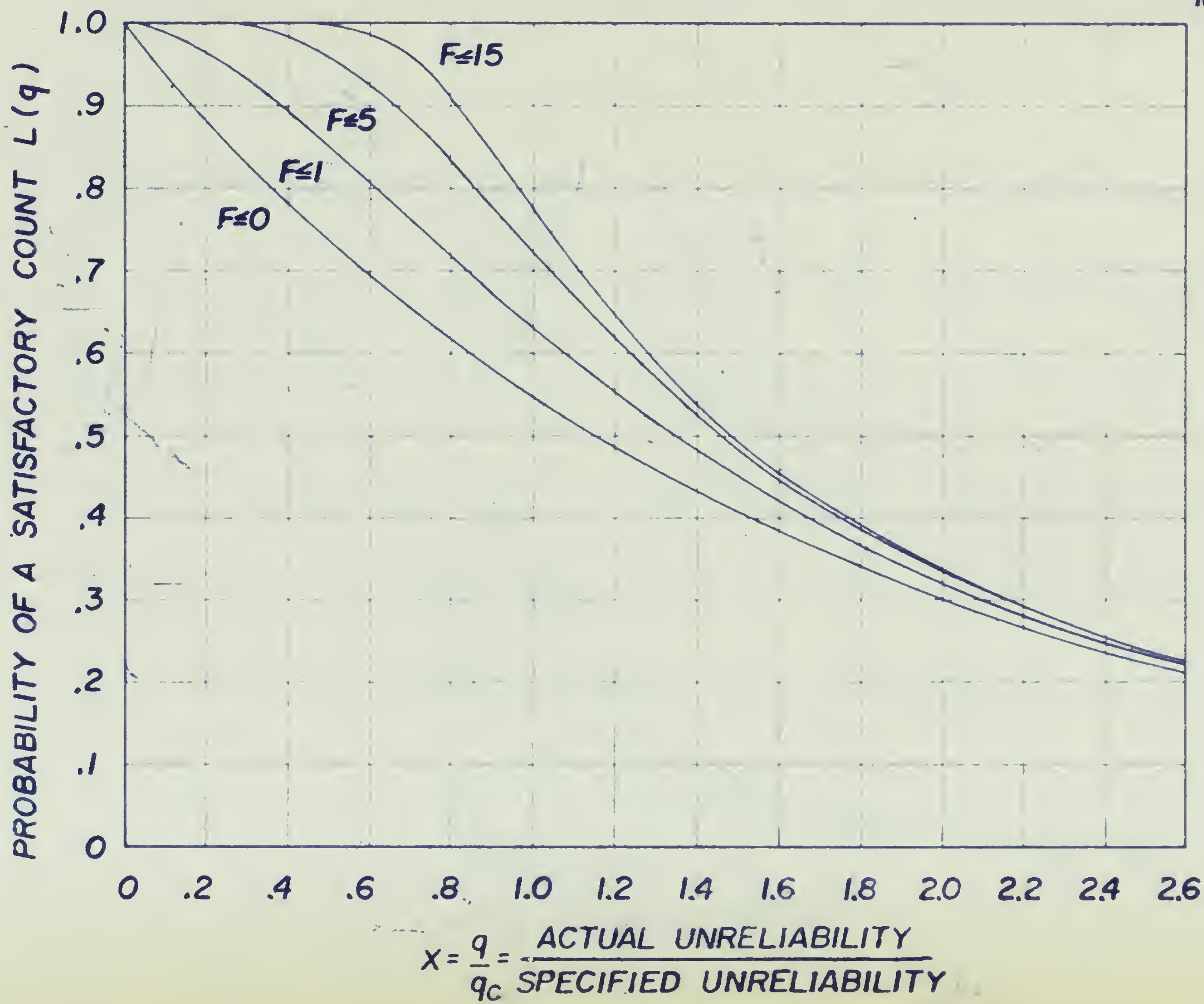


FIGURE 5.23

OPERATING CHARACTERISTIC CURVES AT A 90%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: A=B=O)

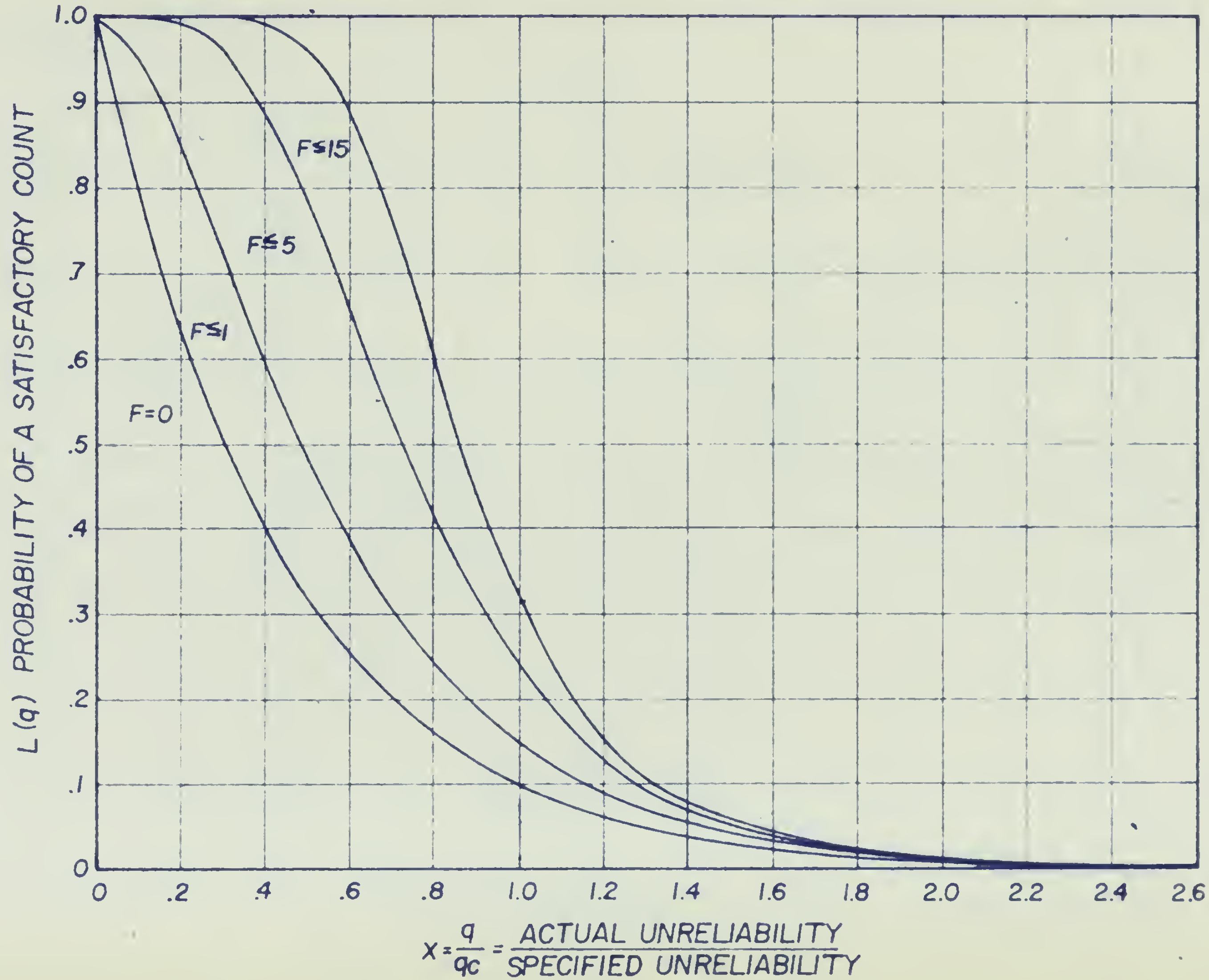
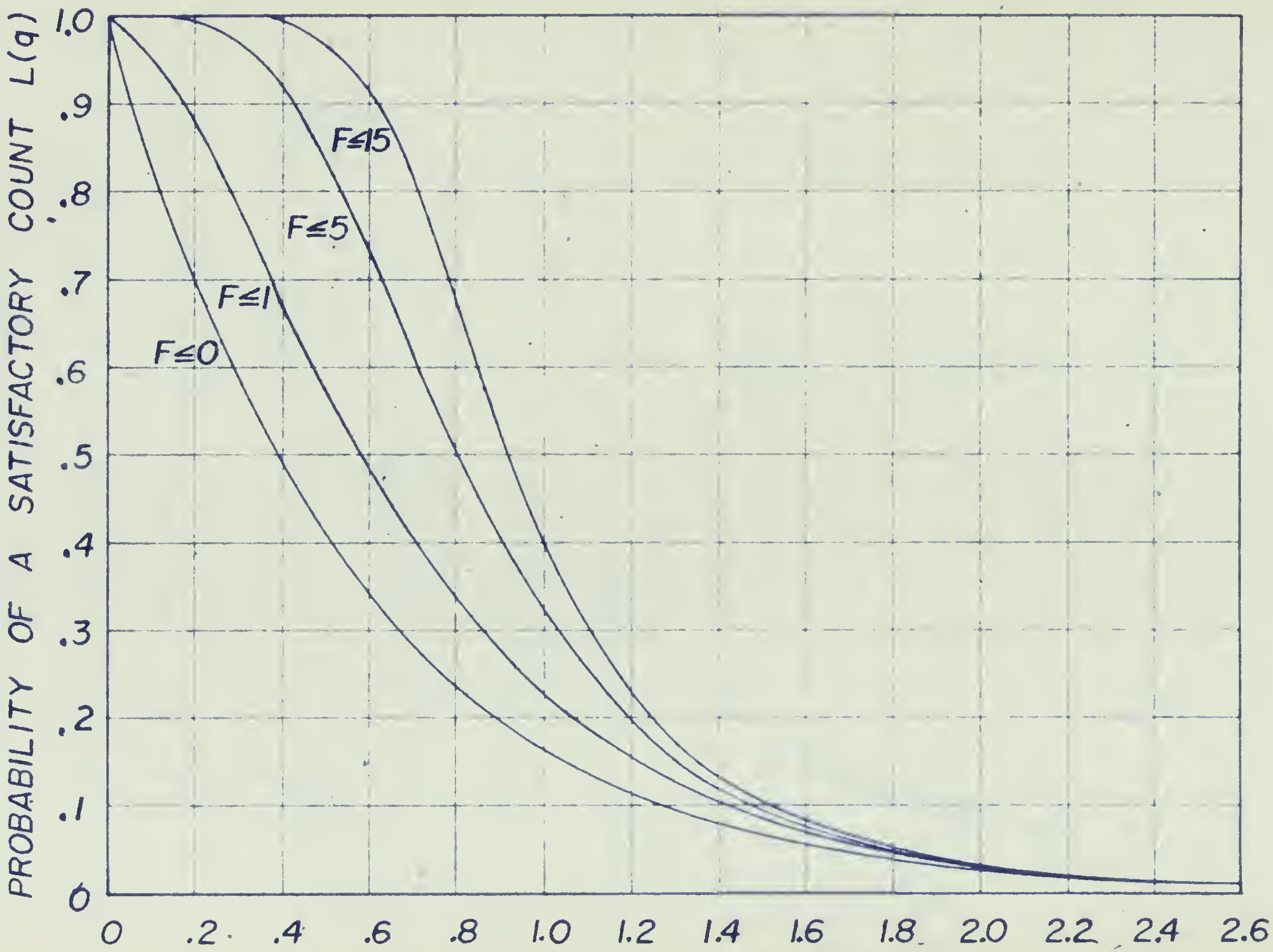


FIGURE 5.31

OPERATING CHARACTERISTIC CURVES AT 90%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS $A=\frac{1}{2q_c}, B=0$)



$$X=\frac{q}{q_c}=\frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.32

OPERATING CHARACTERISTIC CURVES AT A 90% CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A = \frac{1}{q_c}$, $B = 0$)

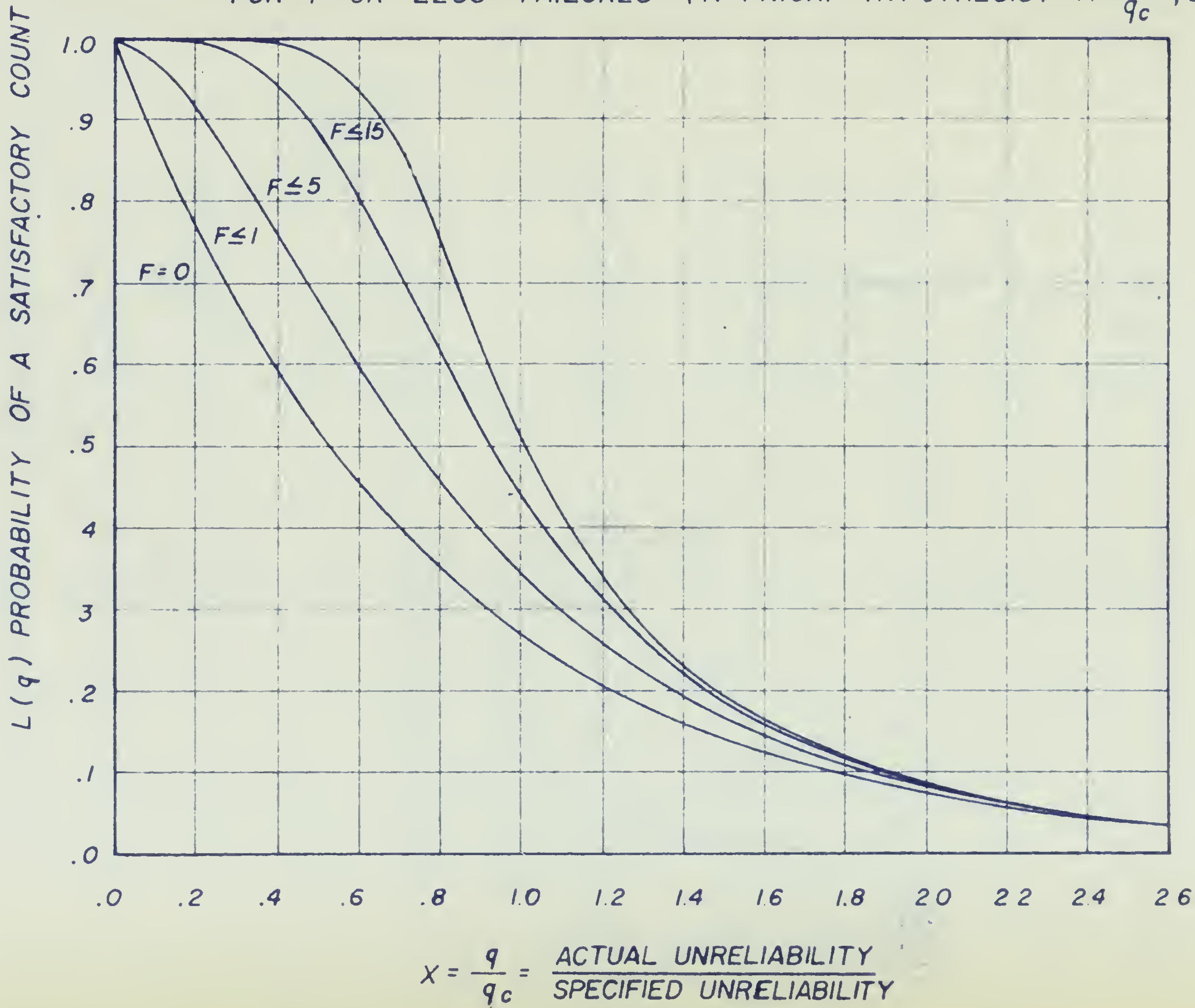
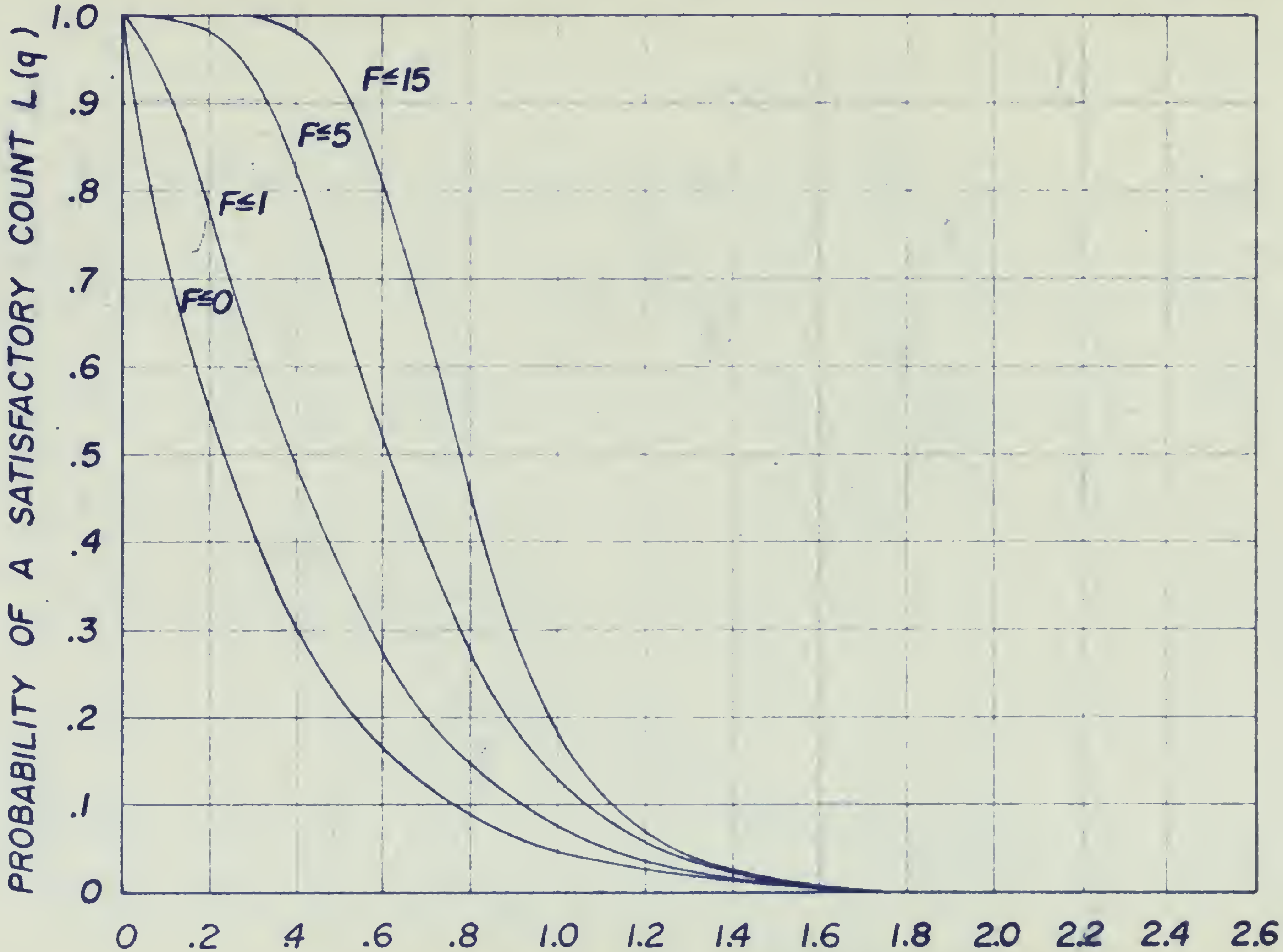


FIGURE 5.33

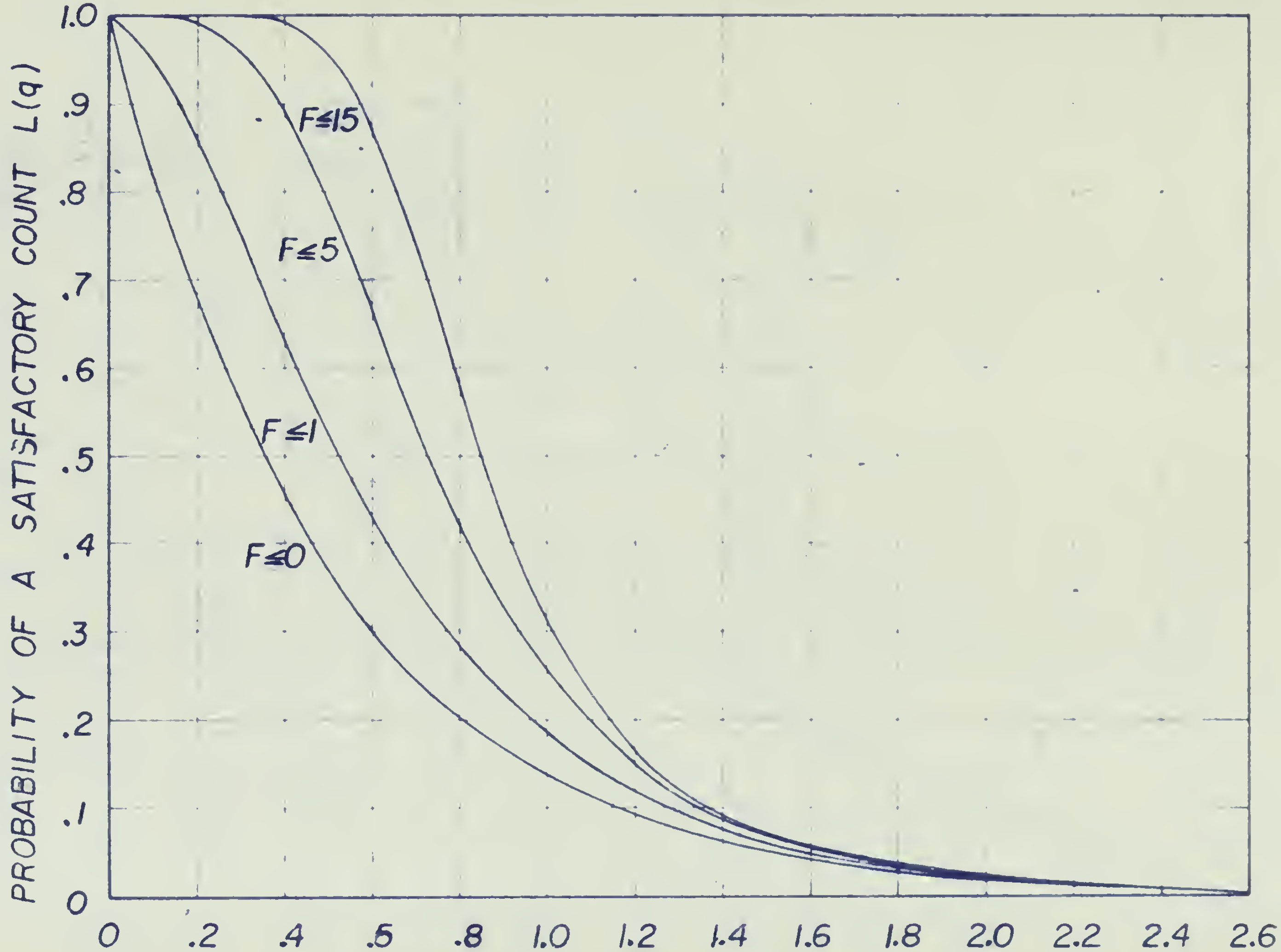
OPERATING CHARACTERISTIC CURVES AT 95% CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS:, A=B=O)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILIY}}$$

FIGURE 5.41

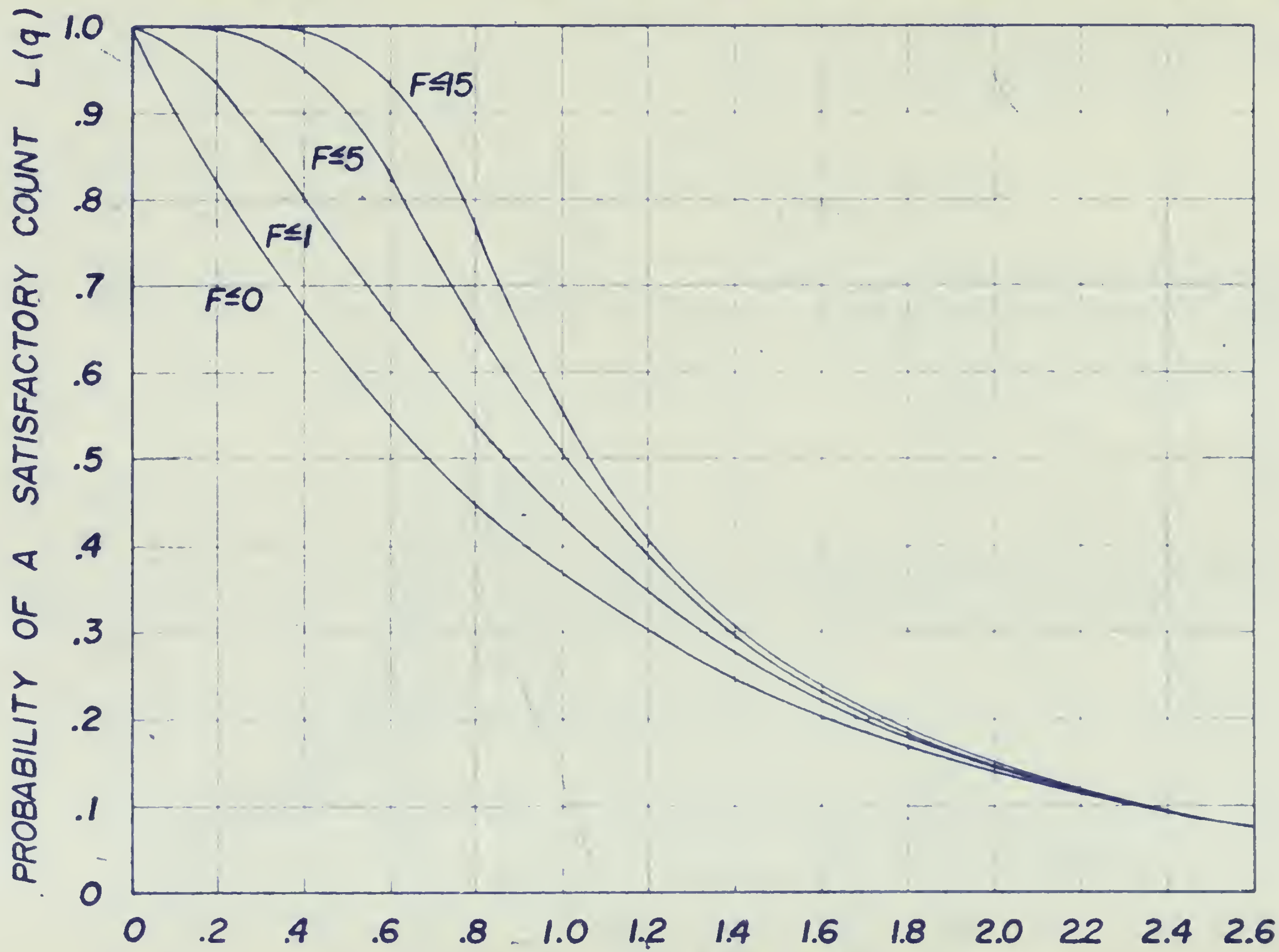
OPERATING CHARACTERISTIC CURVES AT 95 %
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A=\frac{1}{q_c}, B=0$)



$$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$$

FIGURE 5.42

OPERATING CHARACTERISTIC CURVES AT 95%
CONFIDENCE FOR F OR LESS FAILURES (A PRIORI HYPOTHESIS: $A=\frac{2}{q_c}, B=0$)



$X = \frac{q}{q_c} = \frac{\text{ACTUAL UNRELIABILITY}}{\text{SPECIFIED UNRELIABILITY}}$

FIGURE 5.43

CURVES FOR DETERMINING EXPECTED NUMBER OF TESTS (N) AT A 90% CONFIDENCE (A PRIORI HYPOTHESIS: A = B = 0)

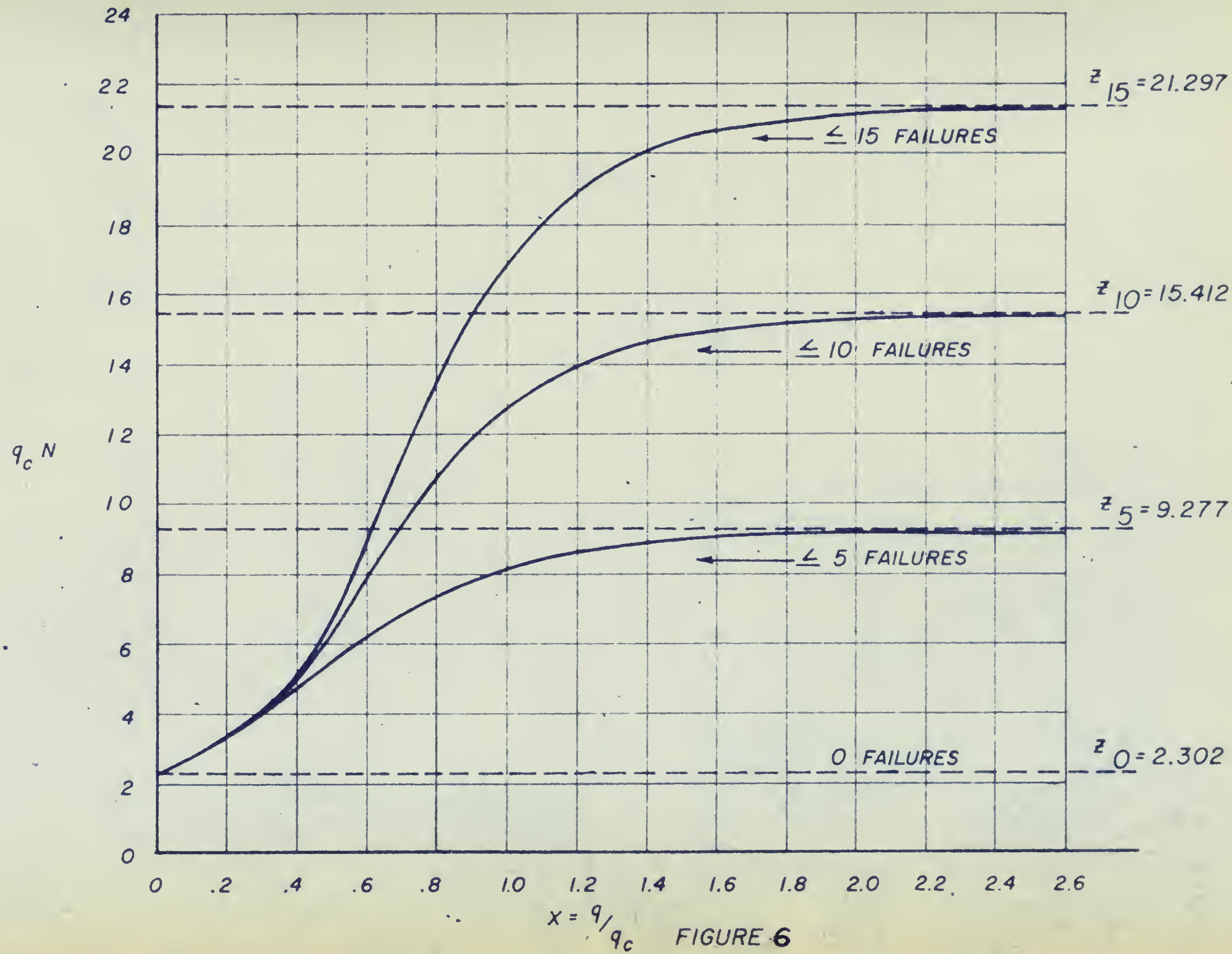


FIGURE 6

CURVES FOR DETERMINING EXPECTED NUMBER OF TESTS (N) AT A 90% CONFIDENCE (A PRIORI HYPOTHESIS: $A=\frac{1}{2q_c}$, $B=0$)

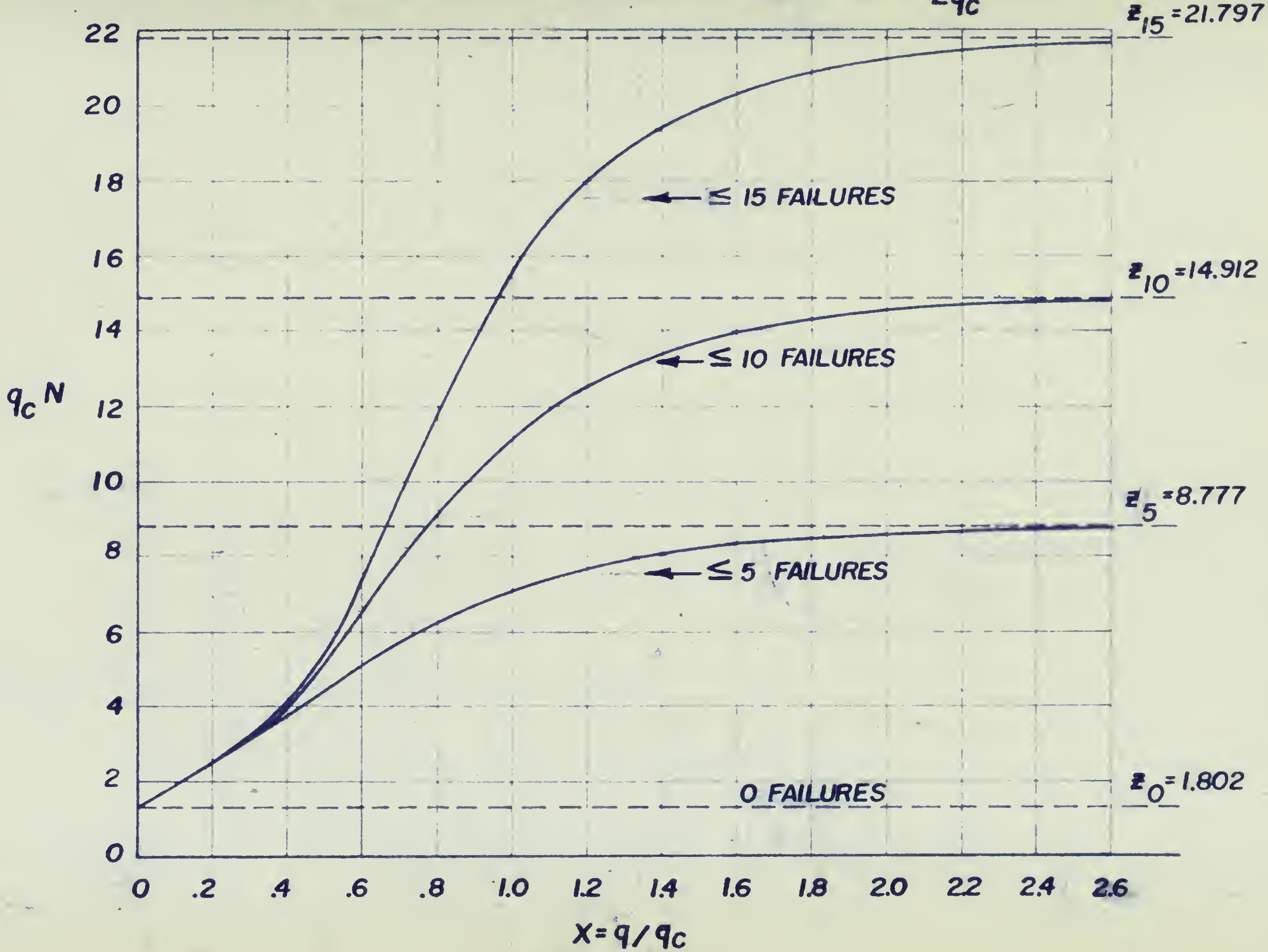


FIGURE 6.1

CURVES FOR DETERMINING EXPECTED NUMBER OF TESTS (N) AT A 90% CONFIDENCE (A PRIORI HYPOTHESIS: $A = \frac{1}{q_c}$, $B = 0$)

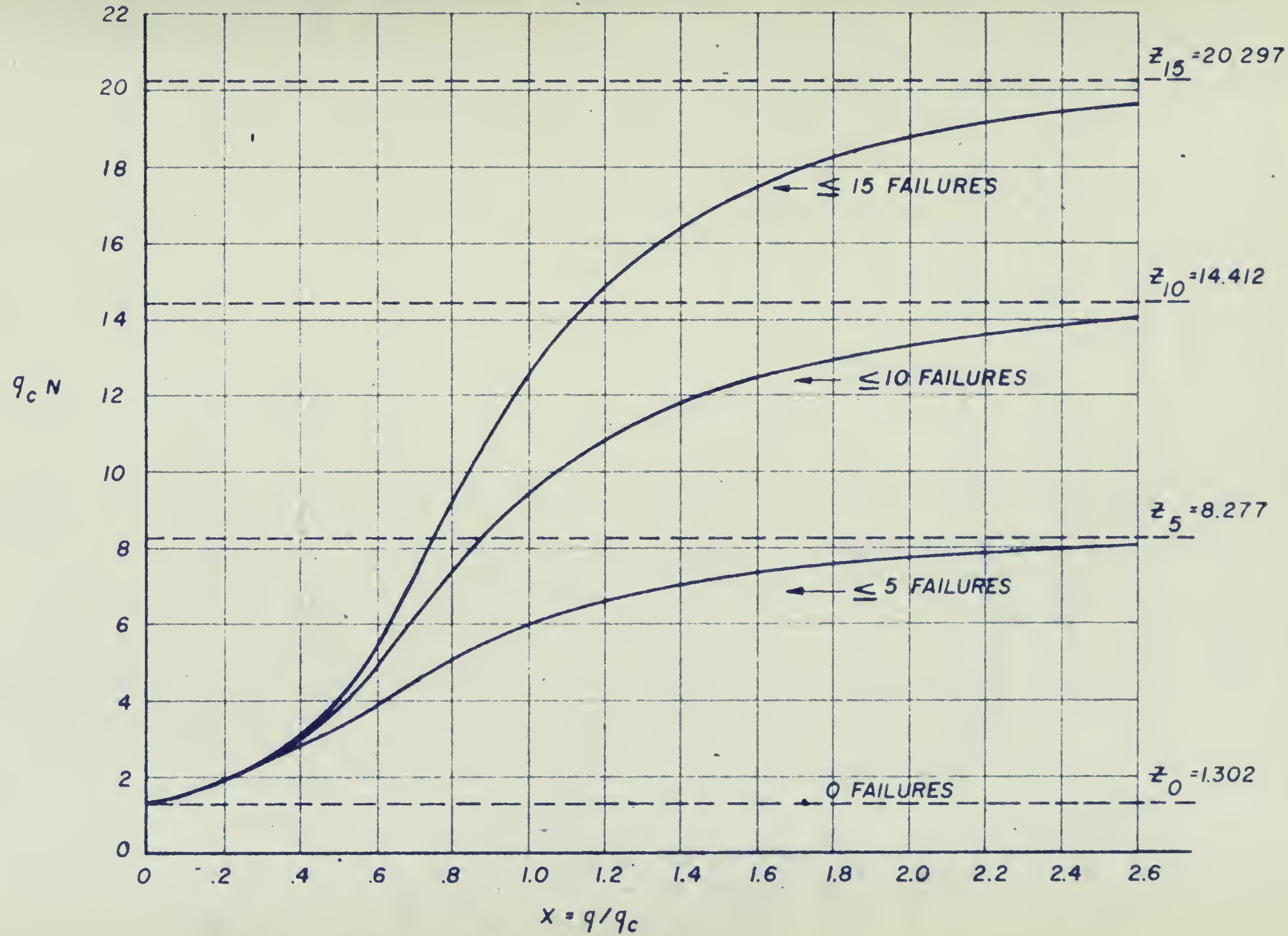


FIGURE 6.2

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2. T. C. Fry, Probability and its Engineering Uses, D. Van Nostrand Company, 1928.

3. A. M. Mood, Introduction to the Theory of Statistics, McGraw-Hill Book Company, Inc., 1950

4. K. Pearson, Tables of the Incomplete Beta Function, Cambridge University Press, 1934.

5. K. Pearson, Tables of the Incomplete Gamma Function, Cambridge University Press, 1934.

APPENDIX I

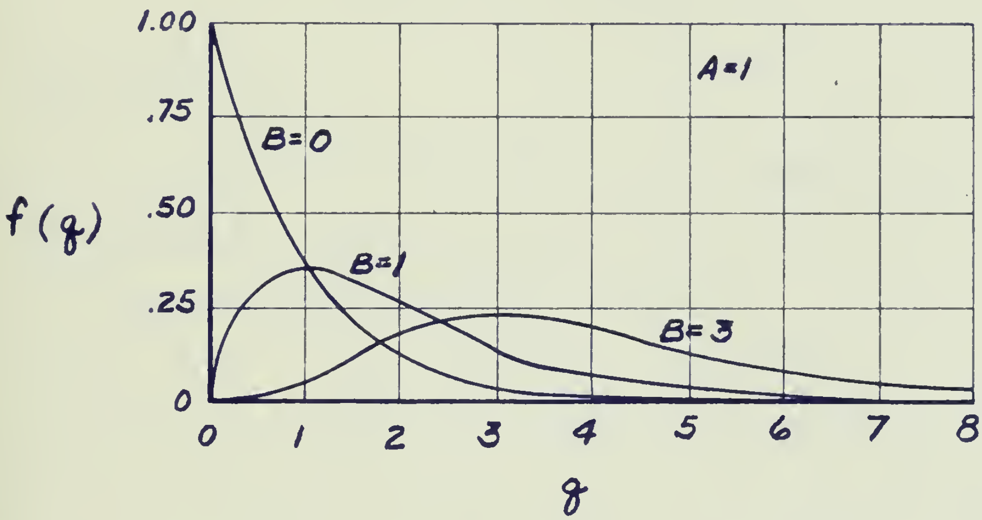
THE GAMMA DISTRIBUTION

The function

$$f(q) = \frac{1}{\left(\frac{1}{A}\right)^{B+1}\Gamma(B+1)} q^{B} e^{-q^{A}} = \frac{A^{B+1} q^{B} e^{-q^{A}}}{\Gamma(B+1)} \qquad q>0$$

$$= 0 \qquad q<0$$

is called the Gamma distribution. This is a two parameter family of distributions, the parameters being A and B. A must be positive and B must be greater than -1. The function is plotted in the Figure below for A = 1 and several values of B.



Changing A merely changes the scale on the two axes, as is evident by examining the form of the function. For example, if A = 10, B = 0 intercept on f(q) axis for q = 0 is 10 and at q = .1, f(q) = 3.678.

To show that the function has a unit area, we shall evaluate the integral:

$$a = \int_0^{\infty} A^{B+1} q^{B} e^{-q^{A}} dq$$

letting $Z = qA$, $dZ = Adq$

$$a = \int_0^\infty z^B e^{-z}\, dz$$

Hence a is a function of B only. If $A > 0$ we may integrate at once by parts to obtain

$$a(B) = B \int_0^\infty z^{B-1} e^{-z}\, dz$$

It then follows recursively that

$$a(B) = B\, a(B-1)$$

If B is a positive integer, we may evaluate a by

$$a(B) = B(B-1)(B-2)\cdots(2)(1)\, B(0)$$

and since

$$B(0) = \int_0^\infty e^{-z}\, dz = 1$$

it is found that

$$a(B) = B!$$

when B is an integer

The function B ! is often denoted by $\Gamma(B + 1)$.

The cumulative distribution function is

$$F(q) = \int_0^{q_c} \frac{A^{B+1}}{\Gamma(B+1)} q^B e^{-qA}\, dq \qquad q>0$$

$$= 0 \qquad q<0$$

When A is a positive integer, the function can be evaluated by successive integration by parts

$$F(q) = 1 - \left[1 + qA + \frac{1}{2!}(qA)^2 + \cdots + \frac{1}{B!}(qA)^B\right] e^{-qA} \qquad q>0$$

with $z = qA$

$$F(z) = 1 - \left[1 + z + \frac{z^2}{2!} + \cdots + \frac{z^B}{B!}\right] e^{-z} \qquad z>0$$

APPENDIX II

THE BETA DISTRIBUTION

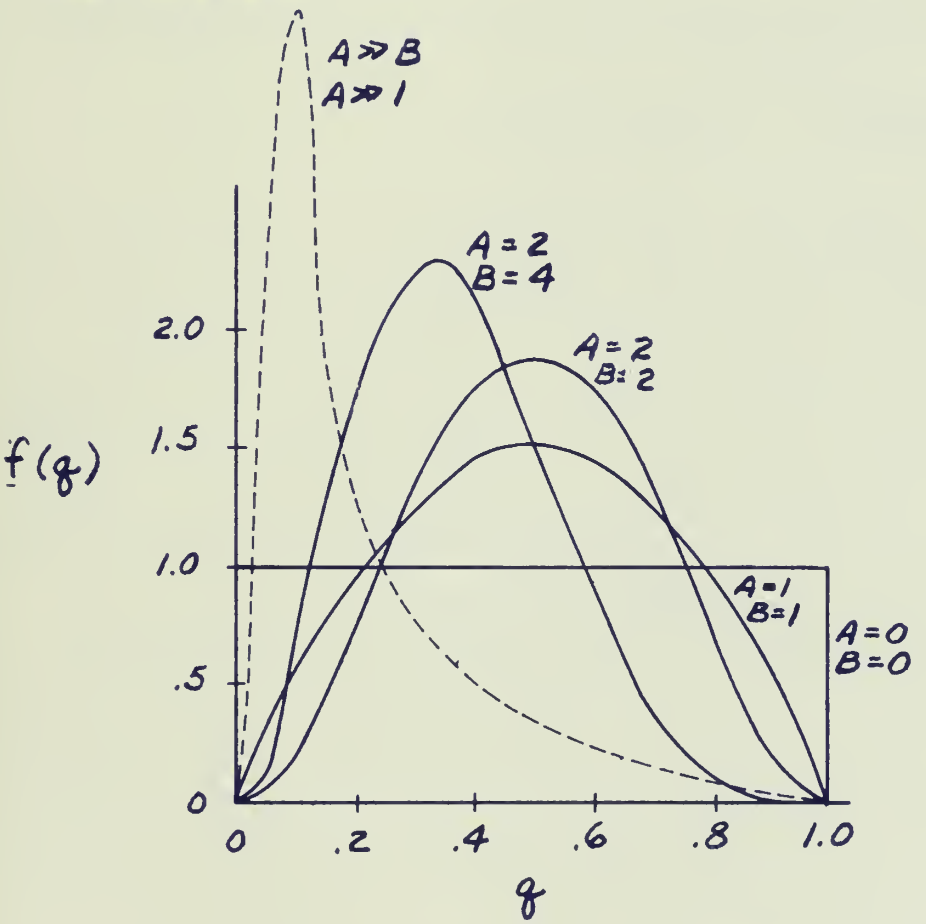
The distribution function

$$f(q) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)}(1-q)^A q^B \qquad 0 < q < 1$$

(1)

$$= 0 \qquad \text{elsewhere}$$

is called a Beta distribution function. The function represents a two-parameter family of distributions. A few examples are plotted in the Figure below.



The distribution becomes the uniform distribution over the unit interval when A = B = 0.

To show that the area under f(q) is unity, we shall compute the integral

$$(2) \qquad \mathcal{A}(A,B) = \int_0^1 (1-q)^A q^B \, dq$$

Clearly, $\mathcal{A}$ will be a function of A and B. One way to demonstrate this is to show that $\mathcal{A}$ is the reciprocal of the constant multiplier in (1). In the discussion of the Gamma function, it is shown that:

$$\Gamma(A+1)\Gamma(B+1) = A!\,B! = \left(\int_0^\infty x^A e^{-x} dx\right)\left(\int_0^\infty y^B e^{-y} dy\right)$$

$$= \int_0^\infty \int_0^\infty x^A y^B e^{-(x+y)} dx\, dy$$

let us make a change of variable by letting $\mu = \dfrac{y}{x+y}$

from which

$$y = \frac{\mu x}{1-\mu}$$

$$dy = \frac{x\, d\mu}{(1-\mu)^2}$$

since $0 \leq \mu \leq 1$ the integral becomes

$$\Gamma(A+1)\Gamma(B+1) = \int_0^\infty \int_0^1 \left(\frac{\mu x}{1-\mu}\right)^B x^A e^{-\frac{x}{(1-\mu)}} \frac{x}{(1-\mu)^2} d\mu\, dx$$

Now let $x \rightarrow v$ by the substitution

$$x = (1-\mu)v$$

$$dx = (1-\mu)dv$$

from which

$$\Gamma(A+1)\Gamma(B+1) = \int_0^\infty \int_0^1 u^B (1-u)^A v^{A+B+1} e^{-v} \, du \, dv$$

$$= \left(\int_0^\infty v^{A+B+1} e^{-v} dv \right) \left(\int_0^1 u^B (1-u)^A du \right)$$

$$= (A+B+1)! \int_0^1 u^B (1-u)^A du$$

$$\therefore \frac{\Gamma(A+1)\Gamma(B+1)}{\Gamma(A+B+2)} = \int_0^1 (1-u)^A u^B du$$

This shows that α (A, B) is the reciprocal of the constant multiplier for (1). α (A-1, B-1) is called the <u>Beta Function</u> in the literature.

The cumulative distribution, the <u>incomplete Beta function</u>, is:

$$F(q) = 0 \qquad q < 0$$

$$= \int_0^{q_c} \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} (1-q)^A q^B dq \qquad 0 < q < 1$$

$$= 1 \qquad q > 1$$

Extensive tabulations of this function can be found in Karl Pearson's <u>"Table of the Incomplete Beta Function"</u>, Cambridge University Press, London, 1932.

The moment generating function for the Beta distribution does not have a simple form. This fact makes the use of the Beta distribution for determining system reliability through combining the component distributions very impractical. It will be noted that the Gamma distribution is ideally suited for the purpose.

APPENDIX III

REPLACEMENT OF BETA DISTRIBUTION BY THE GAMMA DISTRIBUTION

The a posteriori distribution of q was determined to be the Beta distribution function $f_N(q)$, where

$$f_N(q) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)}(1-q)^A q^B$$

From Stirling's formula

$$m! \sim \sqrt{2\pi}\; m^{m+\frac{1}{2}} e^{-m}$$

we can write

$$\frac{\Gamma(A+B+2)}{\Gamma(A+1)} = \frac{(A+B+1)!}{A!} = \frac{\sqrt{2\pi}\,(A+B+1)^{(A+B+\frac{3}{2})} e^{-(A+B+1)}}{\sqrt{2\pi}\; A^{(A+\frac{1}{2})} e^{-A}}$$

$$= \frac{(A+B+1)^{A+\frac{1}{2}}(A+B+1)^{B+1}}{A^{A+\frac{1}{2}}} e^{-(B+1)}$$

$$= (A+B+1)^{B+1}\left(1+\frac{B+1}{A}\right)^{A+\frac{1}{2}} e^{-(B+1)}$$

Recalling that $\lim_{A\to\infty}\left(1+\frac{X}{A}\right)^A = e^X$

it follows that

$$\frac{\Gamma(A+B+2)}{\Gamma(A+1)} = A^{B+1}\left(1+\frac{B+1}{A}\right)^{B+1} e^{(B+1)} e^{-(B+1)} \sim A^{B+1}$$

Letting $q = \frac{Z}{A}$, $dq = \frac{dZ}{A}$

$$f_N(q)\,dq = \frac{A^{B+1}\left(1-\frac{Z}{A}\right)^A\left(\frac{Z}{A}\right)^B \frac{dZ}{A}}{\Gamma(B+1)}$$

In the limiting case, as $A \rightarrow \infty$, this reduces to a single parameter family, with B as the parameter

$$\alpha(z_c) = \int_0^{z_c} \frac{e^{-z} z^B}{\Gamma(B+1)} dz$$

where

$$\mu_z = B + 1$$

$$\sigma_z = \sqrt{B+1}$$

which is a form of the Gamma distribution that is discussed in another section of this Appendix.

One advantage resulting from the replacement of the Beta distribution with the Gamma distribution, is the ease of combining several distributions of the same kind. While several Beta distributions do not combine to form another Beta, the sum of several independent Gamma distributions gives another Gamma.

This is evidenced by considering the moment generating function of the Gamma distribution

$$M_z(\theta) = \int_0^{\infty} e^{\theta z} \frac{e^{-z} z^B}{\Gamma(B+1)} dz$$

$$= \int_0^{\infty} \frac{z^B e^{-(1-\theta)z}}{\Gamma(B+1)} dz$$

$$= \frac{1}{(1-\theta)^{B+1}} \int_0^{\infty} \frac{(1-\theta)^{B+1}}{\Gamma(B+1)} z^B e^{-(1-\theta)z} dz$$

$$= \frac{1}{(1-\theta)^{B+1}}$$

if $Z = \sum_{i=1}^{m} Z_i = Z_1 + Z_2 + \ldots + Z_m$, where Z_i has incomplete Gamma distributions with single parameter B_i, it follows that

$$M'_Z(\theta) = \prod_{i=1}^{m} M_{Z_i}(\theta) = \frac{1}{(1-\theta)^{\sum_i^m (B_i+1)}}$$

which is of the same form as M_z

APPENDIX IV

CALCULATION OF THE MEAN, MODE, AND VARIANCE OF THE BETA DISTRIBUTION

The equation of the Beta distribution being

$$f_N(q) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)}(1-q)^A q^B$$

the following calculations determine

(1) Mean:

$$\mu_q = E(q) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)}\int_0^1 (1-q)^A q^{B+1}\,dq$$

$$= \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} \cdot \frac{\Gamma(A+1)\Gamma(B+2)}{\Gamma(A+B+3)}\int_0^1 \frac{\Gamma[A+(B+1)+2]}{\Gamma(A+1)\Gamma[(B+1)+1]} q^{B+1}(1-q)^A\,dq$$

$$= \frac{\Gamma(A+B+2)\Gamma(B+2)}{\Gamma(B+1)\Gamma(A+B+3)} = \frac{(A+B+1)!\,(B+1)!}{B!\,(A+B+2)!}$$

$$= \frac{B+1}{A+B+2}$$

(2) Variance:

$$\sigma_q^2 = E(q^2) - \left[E(q)\right]^2$$

$$E(q^2) = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)} \int_0^1 q^{B+2}(1-q)^A \, dq$$

$$= \frac{\Gamma(A+B+2)\Gamma(A+1)\Gamma(B+3)}{\Gamma(A+1)\Gamma(B+1)\Gamma(A+B+4)} \int_0^1 \frac{\Gamma(A+B+4)}{\Gamma(A+1)\Gamma(B+3)} q^{B+2}(1-q)^A \, dq$$

$$= \frac{(A+B+1)!\,(B+2)!}{B!\;(A+B+3)!} = \frac{(B+2)(B+1)}{(A+B+3)(A+B+2)}$$

$$\left[E(q)\right]^2 = \frac{(B+1)^2}{(A+B+2)^2}$$

$$\sigma_q^2 = \frac{(B+2)(B+1)}{(A+B+3)(A+B+2)} - \frac{(B+1)^2}{(A+B+2)^2}$$

$$= \frac{(A+1)(B+1)}{(A+B+3)(A+B+2)^2}$$

(3) Mode: ν_q

$$f_o(q) = K(1-q)^A q^B \qquad K = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)}$$

$$\frac{d f_o(q)}{dq} = K\left[-A(1-q)^{A-1} q^B + (1-q)^A B q^{B-1}\right]$$

$$\frac{d f_o(q)}{dq} = 0$$

$$A(1-q)^{A-1} q^B = B(1-q)^A q^{B-1}$$

$$q = \frac{B}{A}(1-q)$$

$$q = \frac{B}{A\left(1+\frac{B}{A}\right)}$$

$$\therefore \quad \nu_q = \frac{B}{A+B}$$

APPENDIX V

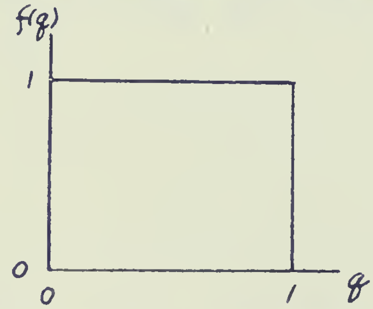
BAYES' THEOREM

An understanding of the use of Bayes' theorem is important to the complete understanding of the proposed method. A single example illustrates how the Bayes' relationship is used to modify an a priori distribution to give an a posteriori distribution.

A statement of Bayes' theorem as used in this paper is:

$$f_N(q) = \frac{f_0(q)\, C_S^N\, p^S q^F}{\int_0^1 \left[f_0(q)\, C_S^N\, p^S q^F \right] dq}$$

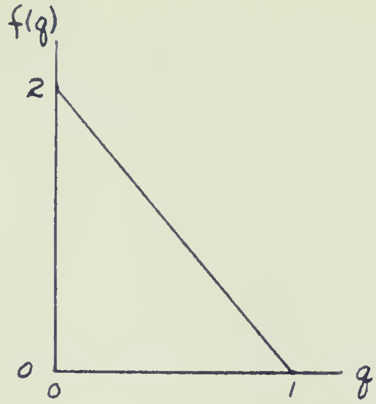
Let one assume that the a priori distribution $f_0(q)$ is uniform between 0 and 1. It may be pictured as follows:



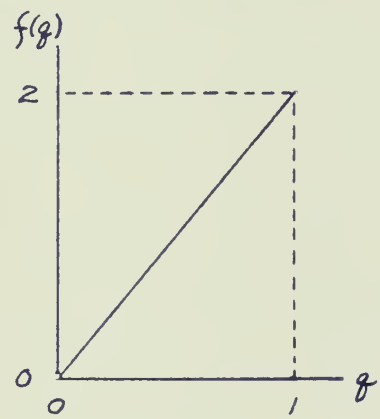
Let one further assume that after one success has been observed, we apply Bayes' formula:

$$f_N(q) = \frac{(1)(p)}{\int_0^1 (1)(p)\, dq} = \frac{(1-q)}{\int_0^1 (1)(1-q)\, dq} = 2(1-q)$$

This a posteriori distribution may be pictured as:



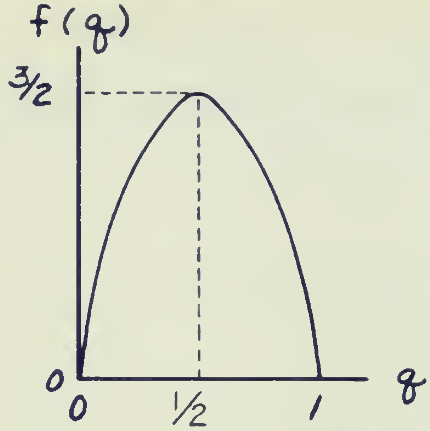
From this one observation, the Bayes' relationship has modified the a priori distribution from uniform to a bias in favor of the value 0. This is an intuitively reasonable result since no failure was observed. If the single observation had been a failure, the a posteriori would have been biased to q = 1 like this:



If the outcomes of two observations were one success and one failure, the a posteriori distribution function would be:

$$f_N(q) = \frac{(1)(1-q)(q)}{\int_0^1 (1)(1-q)(q)\,dq} = 6\left(q-q^2\right)$$

which can be pictured as follows:



This a posteriori distribution function is a parabola, having been modified from the a priori uniform distribution. That the value $q = \frac{1}{2}$ is most heavily biased is reasonable, since we have observed an equal number of successes and failures.

APPENDIX VI

DETERMINATION OF VALUES OF Z

The Tables of the Incomplete Γ-Function are used to determine the value of z for any given confidence level and any given number of failures.

Equivalent notation required for use of the table is as follows:

(1) $u = \dfrac{z}{\sqrt{p + 1}}$

(2) $p = B$ where B = number of failures

(3) $I(u, p)$ = confidence level in decimals

As an example of use of the Γ Function tables, consider the determination of z given 5 failures and a 90% confidence level. With $p = 5$, find the value of u corresponding to the tabular value $I(u, p) = .9000000$. Interpolation is necessary to find $u = 3.7873$. Using the relationship given by (1), one finds:

$$z = u\sqrt{p + 1}$$

$$= (3.7873)\ (2.4495)$$

$$= 9.277$$

In the plotting of the O. C. curves, three sets of assumed initial values of A were investigated. These assumed values of A were introduced by subtracting either $\frac{1}{2}$, 1, or 2 from the value of z, when $A = 0$.

Consider the case with $A = 0$ for a 90% confidence level and failures equal to two. From Table 1 one finds that $z = q_c A \geqslant 5.326$. However, if the assumed value for $A = 1/q_c$, then the a posteriori expression becomes

$$A = S_i + \frac{1}{q_c}$$

and multiplying both sides of the equation results in

$$q_c A = q_c \left(S_i + \frac{1}{q_c}\right) = q_c S_i + 1 \geqslant 5.326$$

or

$$q_c S_i \geqslant 4.326$$

Therefore, when the a priori assumption is $A = \frac{1}{q_c}$, the z used in calculations is the value of z for A = 0 minus one.

In a similar manner, when the assumption is $A = \frac{1}{2q_c}$ or $A = \frac{2}{q_c}$, the assumption is effected by subtracting ½ or 2, respectively, from the value of z when A = 0.

Appendix VIIcontains Table 1 which presents the values of z when A = 0. The values listed are for the 50%, 70%, 80%, 90% and 95% confidence levels and 0 to 15 failures inclusive.

APPENDIX VII

TABLE 1

Values of Z (A = 0)

Failures	Confidence Levels 50	70	80	90	95
0	0.694	1.204	1.601	2.302	2.996
1	1.678	2.440	2.995	3.891	4.746
2	2.674	3.617	4.243	5.326	6.298
3	3.672	4.673	5.518	6.684	7.756
4	4.671	5.892	6.730	7.996	9.154
5	5.669	7.008	7.908	9.277	10.514
6	6.668	8.116	9.078	10.535	11.845
7	7.670	9.211	10.234	11.775	13.153
8	8.664	10.303	11.382	12.999	14.436
9	9.669	11.384	12.520	14.208	15.710
10	10.671	12.472	13.659	15.412	16.961
11	11.670	13.549	14.780	16.517	18.214
12	12.671	14.626	15.899	17.786	19.445
13	13.670	15.696	17.018	18.963	20.674
14	14.671	16.768	18.126	20.128	21.894
15	15.670	17.835	19.246	21.297	23.102

APPENDIX VIII

DEVELOPMENT OF $L(q)$ EXPRESSION

A curve determined by a given value of q_c can be plotted on a graph of successes vs. failures. Then for each value of Failures, i, there exists a number of successes, A_i, that is required to demonstrate the specified q_c.

A "walk" can intercept this q_c curve at only integral values of i. Let $L(q)_i$ be the probability of hitting the q_c curve at i failures not having hit the curve before i failures.

$L(q)$ is the sum of the $L(q)_i$ terms which are as follows:

$$L(q)_0 = p^{A_0}$$

$$L(q)_1 = \left[C_1^{A_0} q p^{A_0-1}\right] p^{A_1-A_0+1} = C_1^{A_0} q p^{A_1} = A_0 q p^{A_1}$$

$$L(q)_2 = \left[C_2^{A_0} q^2 p^{A_0-2}\right] p^{A_2-A_0+2} + \left[\left(C_1^{A_0} q p^{A_0-1}\right)\left(C_1^{A_1-A_0+1} q p^{A_1-A_0}\right) p^{A_2-A_1+1}\right]$$

$$L(q)_3 = \left[C_3^{A_0} q^3 p^{A_0-3}\right] p^{A_3-A_0+3} + \left[C_2^{A_0} q^2 p^{A_0-2}\right]\left[C_1^{A_2-A_0+2} q p^{A_2-A_0+1}\right] p^{A_3-A_2+1}$$

$$+ \left[C_1^{A_0} q p^{A_0-1}\right]\left[C_2^{A_1-A_0+1} q^2 p^{A_1-A_0-1}\right] p^{A_3-A_1+2}$$

$$+ \left[C_1^{A_0} q p^{A_0-1}\right]\left[C_1^{A_1-A_0+1} q p^{A_1-A_0}\right]\left[C_1^{A_2-A_1+1} q p^{A_2-A_1}\right] p^{A_3-A_2+1}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Simplifying, these expressions result in:

$$L(q)_0 = p^{A_0}$$

$$L(q)_1 = A_0 q p^{A_1}$$

$$L(q)_2 = \left[\frac{A_0(A_0-1)}{2} + A_0(A_1-A_0+1)\right] q^2 p^{A_2}$$

$$L(q)_3 = \left[A_0(A_1-A_0+1)(A_2-A_1+1) + A_0\frac{(A_1-A_0+1)(A_1-A_0)}{2} + \frac{A_0(A_0-1)(A_2-A_0+2)}{2} + \frac{1}{6}A_0(A_0-1)(A_0-2)\right] q^3 p^{A_3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Let

$$A_i = \frac{Z_i}{q_c}$$

$$X = \frac{q}{q_c}$$

and consider the limiting case as $q_c \longrightarrow 0$.

For the first term, the approach yields

$$L(q)_0 = (1-q)^{A_0} = (1 - Xq_c)^{Z_0/q_c}$$

$$\ln\left[L(q)_0\right] = \frac{Z_0}{q_c} \ln(1 - Xq_c)$$

$$\lim_{q_c \to 0} \ln\left[L(q)_0\right] = \lim_{q_c \to 0} Z_0\left[\frac{-X}{(1-Xq_c)}\right] = -Z_0 X$$

$$\therefore \lim_{q_c \to 0} L(q)_0 = e^{-Z_0 X}$$

Similarly, the successive terms may be considered to yield:

$$
\begin{aligned}
L(q) = {} & e^{-z_0 x} + z_0 x e^{-z_1 x} + \left[z_0 (z_1 - z_0) + \frac{z_0^2}{2} \right] x^2 e^{-z_2 x} \\
& + \left[\frac{1}{6} z_0^3 + \frac{1}{2} z_0^2 (z_2 - z_0) + \frac{1}{2} z_0 (z_1 - z_0)^2 + z_0 (z_1 - z_0)(z_2 - z_1) \right] x^3 e^{-z_3 x} \\
& + \Big[\frac{1}{24} z_0^4 + \frac{1}{6} z_0^3 (z_3 - z_0) + \frac{1}{6} z_0 (z_1 - z_0)^3 + \frac{1}{4} z_0^2 (z_2 - z_0)^2 \\
& \quad + \frac{1}{2} z_0^2 (z_2 - z_0)(z_3 - z_2) + \frac{1}{2} z_0 (z_1 - z_0)^2 (z_3 - z_1) \\
& \quad + \frac{1}{2} z_0 (z_1 - z_0)(z_2 - z_1)^2 + z_0 (z_1 - z_0)(z_2 - z_1)(z_3 - z_2) \Big] x^4 e^{-z_4 x} \\
& + \ldots
\end{aligned}
$$

It is apparent that evaluation of $L(q)$ becomes laborious as we consider more failures. For the results plotted in this paper, an electronic digital computer was used.

Appendix X presents two computing techniques that are peculiarly associated with this problem.

APPENDIX IX

TABLE 2

q_c	Number of Failures	Minimum Number of Successes Required Confidence Level 50	70	80	90	95
.05	0	13.9	24.1	32.0	46.0	59.9
	1	33.6	48.8	59.9	79.8	94.9
	2	53.5	72.4	84.9	106.5	126.0
	3	73.4	95.2	110.4	133.7	155.1
	4	93.4	118.0	134.6	159.9	183.1
	5	113.4	140.2	158.2	185.5	210.3
	6	133.4	162.6	181.6	210.7	236.9
	7	153.4	184.2	204.7	235.5	263.1
	8	173.4	206.0	227.6	260.0	288.7
	9	193.4	227.6	250.4	284.2	314.2
	10	213.4	249.4	273.2	308.2	339.2
	11	233.4	271.0	295.6	330.3	364.3
	12	253.4	293.0	318.0	355.7	388.9
.10	0	6.9	12.0	16.0	23.0	30.0
	1	16.8	24.4	30.0	39.0	47.5
	2	26.8	36.2	42.4	53.3	63.0
	3	36.7	47.6	55.2	66.8	77.6
	4	46.7	59.0	67.3	80.0	91.5
	5	56.7	70.1	79.1	94.8	105.1
	6	66.7	81.3	90.8	105.5	118.5
	7	76.7	92.1	102.3	118.0	131.5
	8	86.5	103.0	113.8	130.0	144.4
	9	96.7	113.8	125.2	142.0	157.1
	10	106.7	124.7	136.6	154.0	169.6
	11	116.7	135.5	147.8	165.5	182.1
	12	126.7	146.5	159.0	177.5	194.5
.15	0	4.6	8.0	10.7	15.4	20.0
	1	11.2	16.3	20.0	26.0	31.7
	2	17.8	24.1	28.3	35.5	42.0
	3	24.5	31.8	36.8	44.6	51.7
	4	31.2	39.3	44.9	53.3	61.1
	5	37.8	46.7	52.7	61.9	70.0
	6	44.5	54.1	60.6	70.3	79.0
	7	51.2	61.4	68.3	78.5	87.7
	8	57.8	68.6	75.9	86.7	96.3
	9	64.5	75.9	83.5	94.8	104.8
	10	71.2	83.2	91.1	102.8	113.1
	11	77.8	90.4	98.6	110.2	121.5
	12	84.5	97.6	106.0	118.6	129.7

TABLE 2 (continued)

q_c	Number of Failures	Minimum Number of Successes Required Confidence Level 50	70	80	90	95
.20	0	3.5	6.0	8.0	11.5	15.0
	1	8.4	12.2	15.0	19.5	23.7
	2	13.4	18.1	21.2	26.6	31.5
	3	18.4	23.8	27.6	33.4	38.8
	4	23.4	29.5	33.7	40.0	45.8
	5	28.4	35.0	39.5	47.4	52.6
	6	33.3	40.6	45.4	52.8	59.2
	7	38.4	46.1	51.2	59.0	65.8
	8	43.3	51.5	56.9	65.0	72.2
	9	48.4	56.9	62.6	71.0	78.6
	10	53.4	62.4	68.3	77.0	84.8
	11	58.4	67.7	73.9	82.8	91.1
	12	63.4	73.1	79.5	88.8	97.2
.25	0	2.8	4.8	6.4	9.2	12.0
	1	6.7	9.8	12.0	15.6	19.0
	2	10.7	14.5	17.0	21.3	25.2
	3	14.7	19.1	22.1	26.7	31.0
	4	18.7	23.6	26.9	32.0	36.6
	5	22.7	28.0	31.6	37.1	42.1
	6	26.7	32.5	36.3	42.1	47.4
	7	30.7	36.8	40.9	47.1	52.6
	8	34.7	41.2	45.5	52.0	57.7
	9	38.7	45.5	50.1	56.8	62.8
	10	42.7	49.9	54.6	61.6	67.8
	11	46.7	54.2	59.1	66.1	72.9
	12	50.7	58.5	63.6	71.1	77.8

APPENDIX X

COMPUTING TECHNIQUES

In computing the terms of the $\mathsf{L}(q)$ expression, two valuable techniques were devised. The "reliability coefficient" refers to the coefficient in Z that is associated with each exponential term in the $\mathsf{L}(q)$ expression.

<u>Term</u> <u>Code</u> <u>Generator</u>:

A term code is a sequence of positive integers $p_1, p_2, p_3, \ldots, p_n$, which, when interpreted in a particular manner, represents the calculations necessary to compute a given term of a reliability coefficient. For example, consider the term:

$$\frac{(Z_0)^1}{1!} \cdot \frac{(Z_1-Z_0)^3}{3!} \cdot \frac{(Z_4-Z_1)^1}{1!} \cdot \frac{(Z_5-Z_4)^1}{1!}$$

which is one of the 32 terms in the reliability coefficient for F=6. The term code for this term is 1, 3, 1, 1 which is merely a listing of the successive powers. The formula for calculating the term is:

$$\frac{(Z_0)^{p_1}}{p_1!} \cdot \frac{(Z_{p_1}-Z_0)^{p_2}}{p_2!} \cdot \frac{(Z_{p_1+p_2}-Z_{p_1})^{p_3}}{p_3!} \cdot \frac{(Z_{p_1+p_2+p_3}-Z_{p_1+p_2})^{p_4}}{p_4!} \cdots\cdot$$

where the .... implies we continue this process until all the p_i's in the code have been used. There are 2^{F-1} term codes associated with a given failure level F, and they represent all the possible permutations of positive integers that sum up to F. An algorithm for generating all the term codes for a given failure level is:

1. The first code consists of one integer equal to F.
2. The last code consists of F integers equal to one.
3. Given any code $p_1, p_2, \ldots, p_{m-1}, p_m, 1, 1, \ldots, 1$ $\quad p_m \neq 1$, the next code is $p_1, p_2, \ldots, p_{m-1}, p_m-1, p_n$.

Rule 3 says subtract one from the last integer not equal to one, then follow this by only one integer. This last integer p_n is naturally always chosen such that the sum of the p's equal F. Example: If F=4, then this algorithm produces the 8 term codes,

4; 3,1; 2,2; 2,1,1;

1,3; 1,2,1; 1,1,2; 1,1,1,1.

A subroutine was developed that would apply this algorithm to an existing term code and change it into the next code if possible. If the code was already in its final form, all ones, the subroutine exits to an alternate address. The term code generating routine allows us to have only one code stored in the machine at any one particular time. If all the codes had to be stored in the machine at the same time, the problem would not have been feasible.

Power Raiser:

A subroutine was developed that raises a floating number to an integral power. The technique used is quite dependent upon the fact that we have the binary representation of the power. If

$$p = b_0+b_1 2+b_2 2^2+b_3 2^3+ \dots +b_n 2^n,$$

where

$$b_i = 0 \text{ or } 1,$$

then $$a^p = a^{b_0} \cdot a^{2b_1} \cdot a^{4b_2} \dots a^{2^n b_n}.$$

Thus we generate the sequence a, a^2, a^4, a^{16}, ..., a^{2n} as far as needed, then multiply together the subset corresponding to $b_i=1$ which gives a^p. For example, to compute 3^{11} write $11=(1011)_2$ then compute 3, $3^2=9$, $34=81$, $3^8=6561$

finally $3^{11}= (6561)(9)(3) = (6561)(27) = 177{,}147.$

This technique saves many multiplications when raising to large integral powers, and is considered to be a significant time saver in the present problem.

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